Reduction of Order Key

1. $y''-y'-2y=0 \quad y_1(t)=e^{-t} \quad y_2(t)=v(t)y_1(t)=ve^{-t}$

$y_2'=v'y_1+vy_1' \quad y_2''=v''y_1+2v'y_1'+vy_1'' \quad y_1'=-e^{-t}$

$v''y_1+2v'y_1'+vy_1''-v'y_1-vy_1'-2vy_1=0$

$v''y_1+v'(2y_1'-y_1)+v\underbrace{(y_1''-y_1'-2y_1)}_{=0}=0$ since y_1 solves original ODE.

$v''y_1+v'(2y_1'-\cancel{y}_1)=0$

$v''e^{-t}+v'(-2e^{-t}-e^{-t})=0 \quad \dfrac{v''e^{-t}-3e^{-t}v'}{e^{-t}}=0 \quad v''-3v'=0$

$\frac{d}{dt}[v']=3[v'] \Rightarrow \int\frac{d[v']}{v'}=\int 3\,dt = \ln v'=3t+C \Rightarrow v'=Ae^{3t}$

$\int v'=\int Ae^{3t} \Rightarrow v=Ae^{3t}$ (ignore $\frac{A}{3}$ since A is unknown) $=v(t)$

$y_2=v(t)y_1=Ae^{3t}\cdot e^{-t}=Ae^{2t}=y_2(t) \Rightarrow c_2e^{2t}$

2. $(x-1)y''-xy'+y=0 \quad y_1(x)=e^x \quad y_2=v(x)y_1(x)=ve^x$

$y_2'=v'y_1+vy_1' \quad y_2''=v''y_1+2v'y_1'+vy_1''$

$y_2'=v'e^x+ve^x \quad y_2''=v''e^x+2v'e^x+ve^x$

$(x-1)(v''e^x+2v'e^x+ve^x)-x(v'e^x+ve^x)+ve^x=0 \quad /e^x$

$(x-1)(v''+2v'+v)-xv'-xv+v=0 \quad (x-1)v''+(2x-2-x)v'+$ $\cancel{xv-v-xv+v}=0$

$v''(x-1)+(x-2)v'=0 \quad \int\frac{v''}{v'}=-\frac{x-2}{x-1}=\int -1+\frac{1}{x-1}\,dx \quad x-1\overline{)\,x-2}$ $\;\;1$ $\;\;-x+1$ $\;\;-1$

$\ln v'=-x+\ln|x-1|=\ln e^{-x}+\ln|x-1|=\cancel{\ln}|e^x(x-1)|$

$\int v'=\int e^{-x}(x-1) \Rightarrow v=-(x-1)e^{-x}-e^{-x}=-xe^{-x}+\cancel{e^{-x}}-\cancel{e^{-x}}=-xe^{-x}=v(x)$

$y_2=v(x)y_1(x)=-xe^{-x}(e^x)=-x \quad y_2=c_2x$

3. $t^2y''+2ty'-2y=0 \quad y_1(t)=t \quad y_2=v(t)y_1(t)=vt \quad y_2'=v't+v$

$y_2''=v''t+2v' \quad t^2(v''t+2v')+2t(v't+v)-2vt=v''t^3+2t^2v'+2t^2v'=0$

3 cont'd.

$v''t^3 + 4t^2v' = 0 \;/t^2 \Rightarrow v''t + 4v' = 0 \quad \int \frac{v''}{v'} = \int -\frac{4}{t} \Rightarrow \ln v' = -4\ln t$

$v' = \int t^{-4} \Rightarrow v = \frac{t^{-3}}{-3} = -\frac{1}{3t^3} \quad y_2 = \left(\frac{1}{t^3}\right)(t) = \frac{1}{-t^2} \quad y_2 = c_2 t^{-2}$

4. $xy'' - y' + 4x^3y = 0 \quad y_1(x) = \sin(x^2) \quad y_2 = v(x)\sin(x^2)$

$y_1' = 2x\cos(x^2)$

$y_1'' = 2\cos(x^2) - 4x^2\sin(x^2)$

$y_2' = v'\sin(x^2) + v \cdot 2x\cos(x^2)$

$y_2'' = v''\sin(x^2) + 4v'x\cos(x^2) + v\left(2\cos(x^2) - 4x^2\sin(x^2)\right)$

$x\left(v''\sin x^2 + 4v'x\cos x^2 + 2v\cos x^2 - 4vx^2\sin x^2\right) - v'\sin x^2 - 2vx\cos x^2$

$+ 4x^3 v\sin x^2 = 0$

$v''(x\sin x^2) + v'(4x^2\cos x^2 - \sin x^2) + v(2\cancel{x\cos x^2} - 4x^3\sin x^3 - 2\cancel{x\cos x^2}$
$\cancel{+4x^3\sin x^2}) = 0$

$\frac{v''(x\cancel{\sin x^2})}{v'} = \frac{\cancel{v'}(\sin x^2 - 4x^2\cos x^2)}{x\sin x^2} \Rightarrow \int \frac{v''}{v'} = \frac{1}{x} - 4x\cot x^2$

$u = x^2 \quad du = 2x\,dx$

$-2\int \cot u\,du$

$-2\ln|\sin u|$

$\ln v' = \ln x - 2\ln|\sin x^2| = \ln\left[\frac{x}{\sin^2 x^2}\right]$

$\int v' = \int x\csc^2 x^2 \qquad u = x^2 \quad du = 2x\,dx$

$v = -\frac{1}{2}\cot x^2 \qquad y_2 = \cot x^2 \cdot \sin x^2 = \frac{\cos x^2}{\cancel{\sin x^2}} \cdot \cancel{\sin x^2} = \cos(x^2)$

$y_2 = c_2\cos(x^2)$.

5. $2x^2y'' - xy' + y = 0 \quad y_1(x) = x^{1/2} \quad y_2(x) = v(x)x^{1/2}$

$y_2' = v'x^{1/2} + \frac{1}{2}vx^{-1/2} \qquad y_2'' = v''x^{1/2} + vx^{-1/2} - \frac{1}{4}vx^{-3/2}$

$2x^2\left(v''x^{1/2} + v'x^{-1/2} - \frac{1}{4}vx^{-3/2}\right) - x\left(v'x^{1/2} + \frac{1}{2}vx^{-1/2}\right) + vx^{1/2} = 0$

$v''(2x^{5/2}) + v'(2x^{3/2} - x^{3/2}) + v\left(\cancel{-\frac{1}{2}x^{1/2} - \frac{1}{2}x^{1/2} + x^{1/2}}\right) = 0$

$v''(2x^{5/2}) + v'(x^{3/2}) = 0 \quad /x^{3/2}$

$v''(2x) = -v' \Rightarrow \int \frac{v''}{v'} = \int -\frac{1}{2x} \Rightarrow \ln|v'| = -\frac{1}{2}\ln|x| + C$

$v' = Ax^{-1/2}$

$\int v' = \int Ax^{-1/2}dx \Rightarrow v = 2Ax^{1/2} \Rightarrow y_2 = x^{1/2} \cdot x^{1/2} = x$

$y(x) = c_1x^{1/2} + c_2x$

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6. $(x^2+1)y'' - 2xy' + 2y = 0 \qquad y_1(x) = x \qquad y_2 = v(x)x$

$y_2' = v'x + v \qquad y_2'' = v''x + 2v'$

$(x^2+1)(v''x + 2v') - 2x(v'x+v) + 2vx = 0$

$v''(x^3+x) + v'(2x^2+2-2x^2) - \cancel{2xv + 2xv} = 0$

$v''(x^3+x) + 2v' = 0 \qquad \int \frac{v''}{v'} \cancel{(x^3+x)} = \int \frac{-2v'}{x(x^2+1)} \Rightarrow \ln v' =$

$\frac{-2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \Rightarrow -2 = A(x^2+1) + x(Bx+C) \qquad x=0$

$-2 = A \qquad x = 1$

$= -\frac{2}{x} + \frac{2x}{x^2+1} \qquad -2 = -2(2) + (B+C)$

$2 = B+C \qquad x=-1$

$\ln v' = -2\ln x + \ln|x^2+1| \qquad -2 = -2(2) - (-B+C)$

$\int v' = \int \frac{x^2+1}{x^2} = \int 1 + \frac{1}{x^2} \qquad 2 = B - C$

$\Rightarrow 4 = 2B \Rightarrow B = 2 \rightarrow C = 0$

$v = x - \frac{1}{x}$

$y_2 = (x - \frac{1}{x})(x) = x^2 - 1$

$y(x) = c_1 x + c_2(x^2-1)$

7. $y'' - (\tan x)y' + 2y = 0 \qquad y_1(x) = \sin x \qquad y_2 = v\sin x$

$y_2' = v'\sin x + v\cos x \qquad y_2'' = v''\sin x + 2v'\cos x - v\sin x$

$v''\sin x + 2v'\cos x - v\sin x - \frac{\sin x}{\cos x}(v'\sin x + v\cos x) + 2v\sin x = 0$

$v''\sin x + v'\left(2\cos x - \frac{\sin^2 x}{\cos x}\right) - v(\cancel{\sin x - \sin x} + 2\sin x) = 0$

$\frac{v''}{v'}\frac{\sin x}{\sin x} = \frac{v'}{v'}\frac{\left(\frac{\sin^2 x}{\cos x} - 2\cos x\right)}{\sin x} \Rightarrow \int \frac{v''}{v'} = \frac{\sin x}{\cos x} - 2\frac{\cos x}{\sin x}$

$\ln v' = -\ln|\cos x| - 2\ln|\sin x|$

$v' = \frac{1}{\cos x \sin^2 x} = \sec x \csc^2 x$

$(1+\cot^2 x)$

$\frac{1}{\cos x}\left(1 + \frac{\cos^2 x}{\sin^2 x}\right) =$

$\frac{1}{\cos x} + \frac{\cos x}{\sin^2 x} =$

$\sec x + \cot x \csc x$

$\int v' = \int \sec x + \cot x \csc x$

$v = \ln|\sec x + \tan x| - \csc x$

$y_2 = \sin x \cdot \ln|\sec x + \tan x| - 1$

$y(x) = C_1 \sin x + C_2\left(\sin x \ln|\sec x + \tan x| - 1\right)$

8 · $t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0$

$y_3 = vt^2 \qquad y_3' = v't^2 + 2vt \qquad y_3'' = v''t^2 + 4v't + 2v$

$y_3''' = v'''t^2 + 6v''t + 6v'$

$t^2(t+3)(v'''t^2 + 6v''t + 6v') - 3t(t+2)(v''t^2 + 4v't + 2v) +$

$$6(1+t)(v't^2 + 2vt) - 6vt^2 = 0$$

$v'''(t^5 + 3t^4) + v''(t^4 + 3t^3 - 3t^4 - 6t^3) + v'(\cancel{6t^3 + 18t^2 - 12t^3 - 24t^2 + 6t^2 + 6t^3})$

$$v(\cancel{-6t^2 - 12t + 12t + 12t^2 - 6t^2}) = 0$$

$v'''(t^5 + 3t^4) + v''(-2t^4 - 3t^3) = 0$

$v''' \dfrac{(t^5 + 3t^4)}{t^3} = v'' \dfrac{(2t^4 + 3t^3)}{t^3} \Rightarrow \dfrac{v'''}{v''}\cancel{(t^2+3t)} = \cancel{\dfrac{v''}{v''}} \dfrac{(2t+3)}{t^2+3t}$

$\int \dfrac{v'''}{v''} = \int \dfrac{2t+3}{t^2+3t} \Rightarrow \cancel{\ln} v'' = \cancel{\ln} |t^2+3t|$

$\int v'' = \int t^2 + 3t \Rightarrow v' = \dfrac{t^3}{3} + \dfrac{3}{2}t^2 + C \Rightarrow v = \dfrac{t^4}{12} + \dfrac{1}{2}t^3 + Ct + D$

$y_3 = \left[\dfrac{1}{12}t^4 + \dfrac{1}{2}t^3 + Ct + D\right]t^2 = \dfrac{1}{12}t^6 + \dfrac{1}{2}t^5 + \underbrace{Ct^3}_{y_2} + \underbrace{Dt^2}_{y_1}$

$y(t) = c_1 t^2 + c_2 y^3 + c_3\left[\dfrac{1}{12}t^6 + \dfrac{1}{2}t^5\right]$ or

$y(t) = c_1 t^2 + c_2 y^3 + c_3 t^5(t+6)$