MAT 201, Exam #1, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Determine if the points $A(2,4,2), B(3,7,-2), C(1,3,3)$ are collinear. If they are, write an equation of the line in symmetric form. If not, explain why they are not. (5 points)

$\vec{AB} = (3-2, 7-4, -2-2) = \langle 1,3,-4 \rangle$

$\vec{AC} = (1-2, 3-4, 3-2) = \langle -1,-1,-1 \rangle$

No, they are not collinear since $\vec{AB}$ and $\vec{AC}$ are not scalar multiples of each other.

2. For the vectors $\vec{a} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}, \vec{b} = \langle -2,-1,5 \rangle$, find the following: (3 points each)

a. $2\vec{a} + 3\vec{b}$

$\langle 2,4,-6 \rangle + \langle -6,-3,15 \rangle = \langle -4,1,9 \rangle$

b. $\|\vec{a}\|$

$\sqrt{1+4+9} = \sqrt{14}$

c. $\|\vec{a} - \vec{b}\|$

$\langle 1,2,-3 \rangle - \langle -2,-1,+5 \rangle = \langle 3,3,-8 \rangle$

$\|\langle 3,3,-8 \rangle\| = \sqrt{9+9+64} = \sqrt{82}$

d. $\vec{a} \times \vec{b}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ -2 & -1 & 5 \end{vmatrix} = (10-3)\hat{\imath} - (5-6)\hat{\jmath} + (-1+4)\hat{k}$$

$7\hat{\imath} + \hat{\jmath} + 3\hat{k}$

e. $\vec{a} \cdot \vec{b}$

$(-2) + (-2) + (-15) = -19$

3. Decompose the vector $\vec{u} = \langle 3,6,-2 \rangle$ into $\vec{u}_{\parallel} = proj_{\vec{v}}\vec{u}$ and $\vec{u}_{\perp} = \vec{u} - \vec{u}_{\parallel}$. (4 points)

where $\vec{v} = \langle 1, 1, 1 \rangle$

$\frac{3+6-2}{3} = \frac{7}{3}$ $\quad u_{\parallel} = \frac{7}{3}\langle 1,1,1 \rangle = \boxed{\langle 7/3, 7/3, 7/3 \rangle}$

$u_{\perp} = \langle 3,6,-2 \rangle - \langle 7/3, 7/3\ 7/3 \rangle = \boxed{\langle 2/3, 11/3, -13/3 \rangle}$

4. Find the volume of the parallelepiped determined by the vectors $\langle 1,2,3 \rangle, \langle -1,1,2 \rangle, \langle 2,1,4 \rangle$. (3 points)

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} = 1(4-2) - 2\underbrace{(-4-4)}_{-8} + 3\underbrace{(-1-2)}_{-3} =$$

$2 + 16 - 9 = \boxed{9}$

5. Find the distance from the point $(4,1,-2)$ to: (4 points each)
   a. The line $\vec{r}(t) = (1+t)\hat{\imath} + (3-2t)\hat{\jmath} + (4-3t)\hat{k}$

$\langle 1,-2,-3 \rangle = \vec{u}$ $\quad Q = (1,3,4)$ $\quad d = \frac{\sqrt{61}}{\sqrt{14}}$

$\vec{PQ} = \langle 3,-2,-6 \rangle$ $\quad \|\vec{u}\| = \sqrt{1+4+9} = \sqrt{14}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & -6 \\ 1 & -2 & -3 \end{vmatrix} = (6-12)\hat{\imath} - (-9+6)\hat{\jmath} + (-6+2)\hat{k}$$

$-6\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$ $\quad \|\vec{PQ} \times \vec{u}\| = \sqrt{36+9+16} = \sqrt{61}$

   b. The plane $3x + 2y + 6z = 5$.

$\langle 3,2,6 \rangle = \vec{n}$ $\quad \|\vec{n}\| = \sqrt{9+4+36} = \sqrt{49} = 7$

$Q = (1, 1, 0)$

$\vec{PQ} = \langle 3,0,-2 \rangle$

$\langle 3,0,-2 \rangle \cdot \langle 3,2,6 \rangle = 9 + 0 - 12 = -3$

$\frac{\|\vec{PQ} \cdot \vec{n}\|}{\|\vec{n}\|} = \frac{3}{7} = \boxed{4}$

6. Determine the kind of surface for each graph (put the graph in standard form if needed), and sketch or describe the graph including the orientation. (3 points each)

a. $x^2 = y^2 + 4z^2$

Cone, opening around x-axis

b. $x = y^2 - z^2$

hyperbolic paraboloid
normal at saddlepoint along x-axis

c. $x^2 - y^2 + z^2 - 2x + 2y + 4z + 2 = 0$

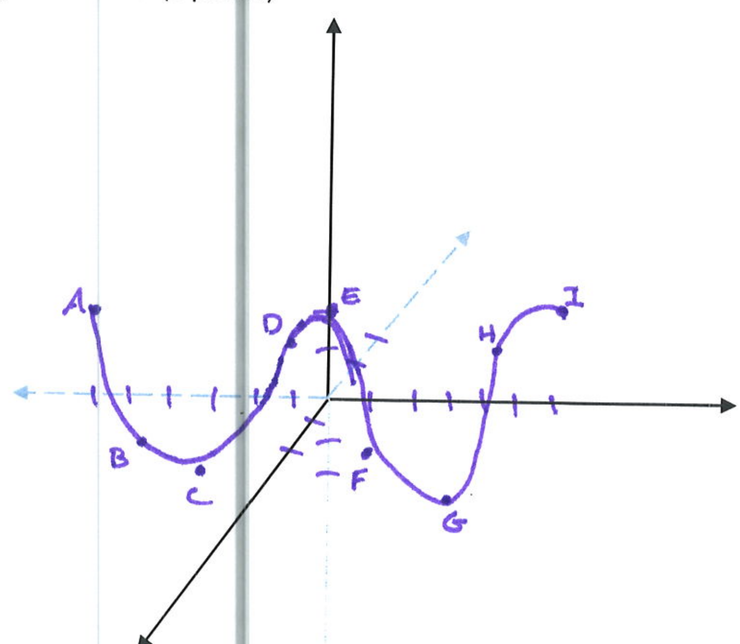
$(x^2 - 2x + 1) - (y^2 - 2y + 1) + (z^2 + 4z + 4) = -2 + 1 - 1 + 4$

$(x-1)^2 - (y-1)^2 + (z+2)^2 = 2$

hyperboloid of one sheet centered at (1,1,-2) wrapped around y-axis
Central radius has circle of radius $\sqrt{2}$

7. Graph the space curve $\vec{r}(t) = 2\sin t\,\hat{\imath} + t\hat{\jmath} + 2\cos t\,\hat{k}$. (8 points)

t	x	y	z	
-2π	0	-6.28	2	A
$-3\pi/2$	2	-4.7	0	B
$-\pi$	0	-3.1	-2	C
$-\pi/2$	-2	-1.57	0	D
0	0	0	2	E
$\pi/2$	2	1.57	0	F
π	0	3.1	-2	G
$3\pi/2$	-2	4.7	0	H
2π	0	6.28	2	I



8. Find a vector function that represents the intersection of $z = \sqrt{x^2+y^2}$ and $z = 1+y$. (6 points)

$(1+y)^2 = (\sqrt{x^2+y^2})^2$

$1+2y+\cancel{y^2} = x^2+\cancel{y^2}$ let $y=t$

$1+2y = x^2$ $\frac{x^2-1}{2} = y$

$1+2t = x^2 \Rightarrow x = \pm\sqrt{1+2t}$

$z = 1+t$

$x = \cos t$

$y = \frac{\cos^2 t - 1}{2} = -\frac{1}{2}\sin^2 t$

option #1

$r_1(t) = \sqrt{1+2t}\,\hat{i} + t\hat{j} + (1+t)\hat{k}$

$r_2(t) = -\sqrt{1+2t}\,\hat{i} + t\hat{j} + (1+t)\hat{k}$

option #2

$r(t) = \cos t\,\hat{i} - \frac{1}{2}\sin^2 t\,\hat{j} + (1 - \frac{1}{2}\sin^2 t)\hat{k}$

there are others as well

9. Find the domain of the function $f(x,y) = \sqrt{y} + \sqrt{15-x^2-y^2}$ write in proper set notation. (4 points)

$x^2+y^2 \le 15$ & $y \ge 0$

$D: \{(x,y) \mid 0 \le y \le \sqrt{15-x^2}\}$

10. Draw at least 5 level curves of the function $f(x,y) = \sqrt{y^2-x^2}$. (8 points)

$0 = \sqrt{y^2-x^2}$

$x^2 = y^2$

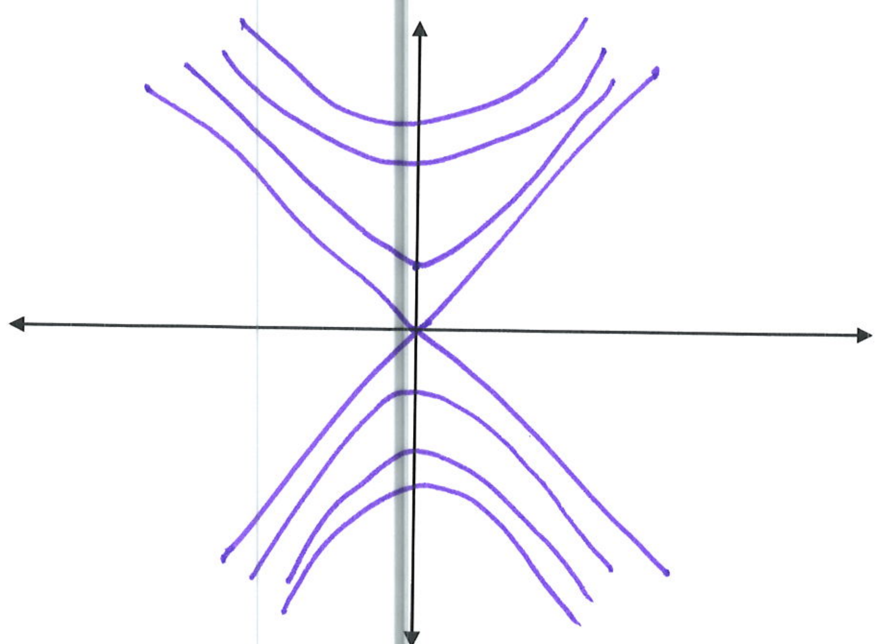
$x = \pm y$

$1 = \sqrt{y^2-x^2}$

$y^2 - x^2 = 1$

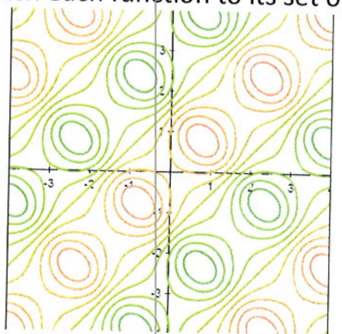
$y^2 = 1+x^2$

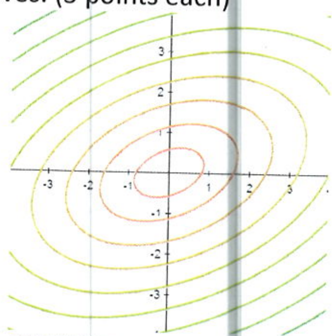
$y = \pm\sqrt{1+x^2}$



11. Match each function to its set of level curves. (3 points each)

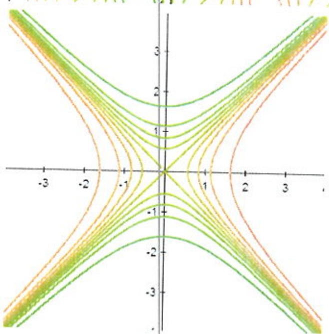
iii

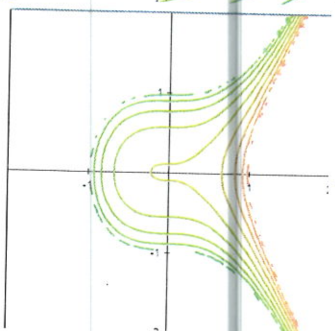
a. 

c. 

i

iv.

b. 

d. 

ii

i. $z = 2 - \sqrt{x^2 - xy + 2y^2}$ c.

ii. $z = \sin^{-1}(x^3 - y^2)$ d.

iii. $z = e^{\sin(x+y)^2} \cos(x - y)$ a.

iv. $z = \tan^{-1}(x^2 - y^2)$ b.

12. Describe the level surface of $f(x, y, z) = y^2 + z^2$ in words. (4 points)

Cylinders, wrapped around x-axis (circular)
w/ larger & larger radii as f increases

13. Find the limit. (5 points each)

a. $\lim\limits_{(x,y)\to(0,0)} \frac{5y^2\cos^2 x}{x^4+y^4}$

$y = kx$

$$\lim_{x\to 0} \frac{5k^2x^2\cos^2 x}{x^4+k^4x^4} = \lim_{x\to 0} \frac{5k^2\cos^2 x \cdot \cancel{x^2}}{x^2\cancel{x^2}(1+k^4)}$$

$$\lim_{x\to 0} \frac{5k^2\cos^2 x}{x^2(1+k^4)} = \infty$$

b. $\lim\limits_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

$kx = y^3$ or $x = ky^3$

$$\lim_{y\to 0} \frac{ky^3\cdot y^3}{k^2y^6+y^6} =$$

$$\lim_{y\to 0} \frac{k\cancel{y^6}}{\cancel{y^6}(k^2+1)} = \lim_{y\to 0} \frac{k}{k^2+1}$$

DNE

since value depends on k

14. For each of the following, convert the equation to the indicated coordinate system and describe or sketch the graph. Solve for the function variable whenever feasible. (4 points each)

a. $x^2 - x + y^2 + z^2 = 1$, cylindrical

$r^2\cos^2\theta - r\cos\theta + r^2\sin^2\theta + z^2 = 1$

$r^2 + r\cos\theta + z^2 = 1$

can't solve for z easily
can stop here.

ellipsoid center at (1/4, 0, 0)

b. $z = 4 - r^2$, rectangular

$z = 4 - x^2 - y^2$

paraboloid opens down vertex at (4,0,0)

c. $x + 2y + 3z = 1$, spherical

$\rho\sin\varphi\cos\theta + 2\rho\sin\varphi\sin\theta + 3\rho\cos\varphi = 1$

$\rho = \dfrac{1}{\sin\varphi\cos\theta + 2\sin\varphi\sin\theta + 3\cos\varphi}$

plane
intercepts at
(1,0,0) (0,1/2,0)
and (0,0,1/3)

d. $\rho = \cancel{\sin\theta\cos\phi}$ $\sin\varphi\cos\theta$, rectangular

$\rho^2 = \rho\sin\varphi\cos\theta$

$x^2 + y^2 + z^2 = x$

$\left(x^2 - x + \frac{1}{4}\right) + y^2 + z^2 = \frac{1}{4}$

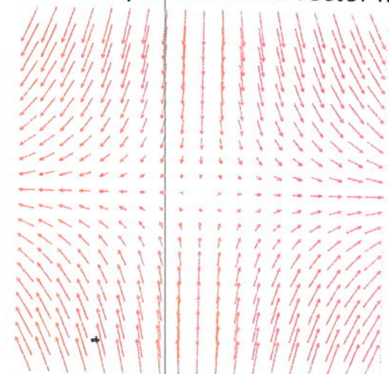
~~ellipsoid~~ – Sphere

Centered at (1/4, 0, 0)

(stretched differently than 14a)

15. Match the equation of the vector field to its graph. (3 points each)

iv

a. 

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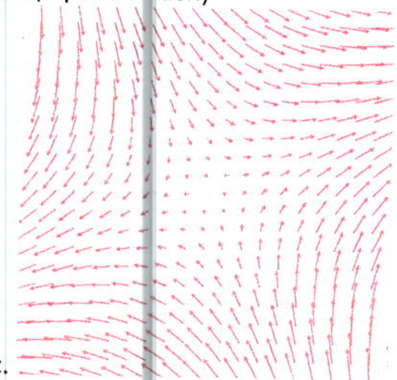
c. 

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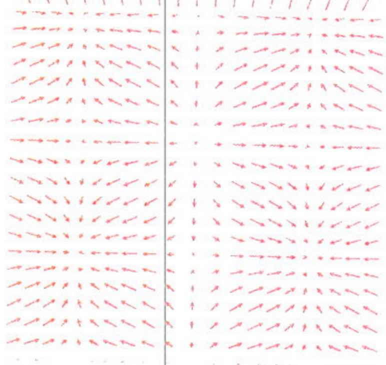
b. 

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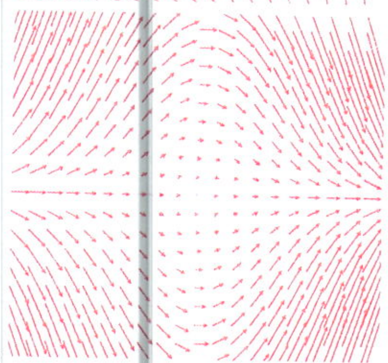
d. 

i. $\vec{F}(x,y) = \sqrt{x^2+y^2}\,\hat{\imath} - xy\hat{\jmath}$ d.

ii. $\vec{F}(x,y) = (x+y)\hat{\imath} + (x-y)\hat{\jmath}$ c

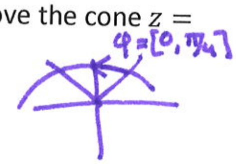
iii. $\vec{F}(x,y) = 4\sin x\,\hat{\imath} - 2\cos y\,\hat{\jmath}$ b.

iv. $\vec{F}(x,y) = x\hat{\imath} - 2y\hat{\jmath}$ a

16. Find a parameterization for the surfaces of the sphere $x^2+y^2+z^2 = 4$ above the cone $z = \sqrt{x^2+y^2}$. Include any boundary conditions. (4 points)

$x^2+y^2+\underbrace{x^2+y^2}_{z^2} = 4$

$r = \sqrt{2}$



$2x^2+2y^2 = 4$

$x^2+y^2 = (\sqrt{2})^2$ intersection

$r(u,v) = 2\sin u\cos v\,\hat{\imath} + 2\sin u\sin v\,\hat{\jmath} + 2\cos u\,\hat{k}$

$0 \le u \le \pi/4 \quad 0 \le v \le 2\pi$

17. Describe the surface described by the parametric surface given by $\vec{r}(u,v) = u\cos v\,\hat{\imath} + u\sin v\,\hat{\jmath} + u^2\hat{k}$. (4 points)

$u^2\cos^2 v + u^2\sin^2 v = u^2$

$x^2 + y^2 = z$ paraboloid

opens up, centered at $(0,0,0)$

Some useful formulas:

$$d = \frac{|\overrightarrow{PQ} \cdot \vec{n}|}{\|\vec{n}\|}$$

$$d = \frac{\|\overrightarrow{PQ} \times \vec{u}\|}{\|\vec{u}\|}$$