MAT 201, Quiz #9, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. For $\vec{r}(t) = \sqrt{2}t\hat{\imath} + e^t\hat{\jmath} + e^{-t}\hat{k}$, find the unit tangent vector and the unit normal vector.

$r'(t) = \sqrt{2}\hat{\imath} + e^t\hat{\jmath} - e^{-t}\hat{k}$ $\quad \|r'(t)\| = \sqrt{2+e^{2t}+e^{-2t}} = \sqrt{(e^t+e^{-t})^2} = e^t+e^{-t}$

$T(t) = \dfrac{\sqrt{2}\hat{\imath} + e^t\hat{\jmath} - e^{-t}\hat{k}}{e^t+e^{-t}}$ or $\dfrac{\sqrt{2}e^t\hat{\imath} + e^{2t}\hat{\jmath} - 1\hat{k}}{e^{2t}+1}$

$T'(t) = \dfrac{\sqrt{2}e^t\hat{\imath} + 2e^{2t}\hat{\jmath}}{e^{2t}+1} + \dfrac{-1\cdot 2e^{2t}}{(e^{2t}+1)^2}\left(\sqrt{2}e^t\hat{\imath} + e^{2t}\hat{\jmath} - 1\hat{k}\right) =$

$\dfrac{(\sqrt{2})(e^{3t}+e^t)\hat{\imath} + 2(e^{4t}+e^{2t})\hat{\jmath} - 2\sqrt{2}e^{3t}\hat{\imath} - 2e^{4t}\hat{\jmath} + 2e^{2t}\hat{k}}{(e^{2t}+1)^2} =$

$\dfrac{(2e^{3t}+\sqrt{2}e^t)\hat{\imath} + 2e^{2t}\hat{\jmath} + 2e^{2t}\hat{k}}{(e^{2t}+1)^2} \Rightarrow$ back page

2. Find the equation of the tangent plane to the function:
   a. $z = x^2 + xy + 3y^2$ at $(1,1,5)$

$F = x^2+xy+3y^2-z$

$\nabla F = (2x+y)\hat{\imath} + (x+6y)\hat{\jmath} - \hat{k}$

$3\hat{\imath} + 7\hat{\jmath} - \hat{k}$

$3(x-1)+7(y-1)-1(z-5)=0$

   b. $\vec{r}(u,v) = u\cos v\,\hat{\imath} + u\sin v\,\hat{\jmath} + v\hat{k}, u = 1, v = \frac{\pi}{3}$

$(1\cdot\frac{1}{2}, 1\cdot\frac{\sqrt{3}}{2}, \pi/3)$

$= (\frac{1}{2}, \frac{\sqrt{3}}{2}, \pi/3)$

$\vec{r}_u = \cos v\,\hat{\imath} + \sin v\,\hat{\jmath} + 0\hat{k} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 0\rangle$

$\vec{r}_v = -u\sin v\,\hat{\imath} + u\cos v\,\hat{\jmath} + 1\hat{k} = \langle -\frac{\sqrt{3}}{2}, 1/2, 1\rangle$

$\vec{r}_u\times\vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1/2 & \sqrt{3}/2 & 0 \\ -\frac{\sqrt{3}}{2} & 1/2 & 1 \end{vmatrix} = (\frac{\sqrt{3}}{2}-0)\hat{\imath} - (1/2+0)\hat{\jmath} + (\frac{1}{4}+\frac{3}{4})\hat{k}$

$\langle \frac{\sqrt{3}}{2}, -1/2, 1\rangle$

$\frac{\sqrt{3}}{2}(x-\frac{1}{2}) - \frac{1}{2}(y-\frac{\sqrt{3}}{2}) + 1(z-\pi/3)=0$

3. For $z = \sin\theta\cos\phi, \phi = s^2 t, \theta = st^2$, use the chain rule to find $\frac{\partial z}{\partial s}$ $= \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial s} + \frac{\partial z}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial s}$

$\frac{\partial z}{\partial \theta} = \cos\theta\cos\varphi = \cos(st^2)\cos(s^2t)$

$\frac{\partial z}{\partial \varphi} = -\sin\theta\sin\varphi = -\sin(st^2)\sin(s^2t)$

$\frac{\partial \theta}{\partial s} = t^2$

$\frac{\partial z}{\partial s} = \cos(st^2)\cos(s^2t)\cdot t^2 - \sin(st^2)\sin(s^2t)\cdot 2st$

$\frac{\partial \varphi}{\partial s} = 2st$

4. Use partial derivatives to find $\frac{\partial z}{\partial y}$ for the implicit function $yz + x\ln y = z^2$.

$F = yz + x\ln y - z^2$

$F_y = z + \frac{x}{y}$

$F_z = y - 2z$

$\frac{\partial z}{\partial y} = -\frac{(z+\frac{x}{y})}{(y-2z)}\cdot\frac{y}{y} = \boxed{\frac{zy+x}{2zy-y^2}}$

5. Find the Jacobian for $x = v + w^2, y = w + u^2, z = u + v^2$.

$$J = \begin{vmatrix} 0 & 1 & 2w \\ 2u & 0 & 1 \\ 1 & 2v & 0 \end{vmatrix} = 0 - 1(0-1) + 2w(4uv) = \boxed{1 + 8uvw}$$

$$\|T'(t)\| = \frac{\sqrt{(2e^{3t}+\sqrt{2}e^{t})^2 + 4e^{4t} + 4e^{4t}}}{(e^{2t}+1)^2} =$$

$$(2e^{3t}+\sqrt{2}e^{t})(2e^{3t}+e^{t}\sqrt{2}) = 4e^{6t} + 2\sqrt{2}e^{4t} + 2\sqrt{2}e^{4t} + 2e^{2t} =$$

$$\frac{\sqrt{4e^{6t} + 4\sqrt{2}e^{4t} + 2e^{2t} + 8e^{4t}}}{(e^{2t}+1)^2} = \frac{\sqrt{4e^{6t} + (4\sqrt{2}+8)e^{4t} + 2e^{2t}}}{(e^{2t}+1)^2}$$

$$\vec{N}(t) = \frac{(e^{2t}+1)^2}{\sqrt{4e^{6t} + (4\sqrt{2}+8)e^{4t} + 2e^{2t}}} \cdot \frac{(2e^{3t}+\sqrt{2}e^{t})\hat{\imath} + 2e^{2t}\hat{\jmath} + 2e^{2t}\hat{k}}{(e^{2t}+1)^2}$$

$$= \frac{(2e^{3t}+\sqrt{2}e^{t})\hat{\imath} + 2e^{2t}\hat{\jmath} + 2e^{2t}\hat{k}}{\sqrt{4e^{6t} + (4\sqrt{2}+8)e^{4t} + 2e^{2t}}} = \frac{\cancel{e^{t}}\left[(2e^{2t}+\sqrt{2})\hat{\imath} + 2e^{t}\hat{\jmath} + 2e^{t}\hat{k}\right]}{\cancel{e^{t}}\sqrt{2e^{4t} + (4\sqrt{2}+8)e^{2t} + 2}}$$

$$= e^{2t}(2e^{4t} + (4\sqrt{2}+8)e^{2t} + 2)$$

$$= \frac{(2e^{2t}+\sqrt{2})\hat{\imath} + 2e^{t}\hat{\jmath} + 2e^{t}\hat{k}}{\sqrt{2e^{4t} + (4\sqrt{2}+8)e^{2t} + 2}}$$