MAT 212, Quiz #11, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Determine if $y_1 = \sin x, y_2 = \cos x$ are orthogonal under the inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$.

$\int_{-\pi}^{\pi} \sin x \cos x \, dx = \frac{1}{2}\sin^2 x \Big|_{-\pi}^{\pi} = 0$

Yes the are orthogonal since their inner product is zero

2. Express each piecewise function in terms of the unit step function.

a. $f(t) = \begin{cases} t^2, & 0 \le t < 2 \\ 1, & t \ge 2 \end{cases}$

$(t-2)^2 + 4(t-2)$

$t^2 =$

$t^2 - 4t + 4 + 4t - 4$

$f(t) = t^2 + 4t$

$1 - t^2$

$1 - (t-2)^2 - 4(t-2)$

$f_2(t) = 1 - t^2 - 4t$

$t^2 + u(t-2)(1-t^2) = f(t)$

MATLAB notation

$f(t) = t^2 + u_2(t)[1 - (t-2)^2 - 4(t-2)]$ (textbook)

b. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ t-2, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$

$t - 1 - t = -1$

$t - 2 - t = -2$

$-t = -(t-3) - 3$

$t - u_1(-1) - (-2)u_2 - u_3 f(t-c)$

$f(t) = t + u_1 + 2u_2 - u_3[-(t-3) - 3]$

textbook

$f(t) = t + u(t-1)(t-1-t) + u(t-2)(t-2-t+1) + u(t-3)(2-t)$

Matlab not.

3. Find the inverse Laplace transform (using the table) of each function.

a. $F(s) = \frac{e^{-2s}}{s^2+s-2} = e^{-2s}\left(\frac{1}{(s+2)(s-1)}\right)$

$u_2(t)\left(-\frac{1}{3}e^{-2t} + \frac{1}{3}e^t\right)$

$f(t) = \frac{1}{3}u_2(t)\left[e^t - e^{-2t}\right]$

$\frac{A}{s+2} + \frac{B}{s-1} =$

$As - A + Bs + 2B = 1$

$A + B = 0 \qquad A = -1/3$

$-A + 2B = 1 \qquad B = 1/3$

$-\frac{1}{3}\left(\frac{1}{s+2}\right) + \frac{1}{3}\left(\frac{1}{s-1}\right)$

b. $F(s) = \frac{s}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$

$As^2 + 4A + Bs^2 + Bs + Cs + C = s$

$A + B = 0 \qquad A = -1/5$

$B + C = 1 \qquad B = 1/5$

$4A + C = 0 \qquad C = 4/5$

$-\frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{1}{5}\left(\frac{s+4}{s^2+4}\right) =$

$-\frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{1}{5}\left(\frac{s}{s^2+4}\right) + \frac{4}{5}\left(\frac{1}{s^2+4}\right)$

$f(t) = -\frac{1}{5}e^{-t} + \frac{1}{5}\cos 2t + \frac{4}{5}\sin 2t$

or $\int_0^t e^{-1(t-\tau)} \cos 2\tau \, d\tau = \int_0^t e^{-\tau}\cos(2(t-\tau))\,d\tau$

4. Use the table to find the Laplace transform of $f(t) = \int_0^t (t-\tau)\cos 2\tau \, d\tau$.

$\int_0^t f(t\ \tau)\ g(\tau)d\tau$

$f(t-\tau) = t \quad g(t) = \cos 2t$

$F(s) = \frac{1}{s^2} \quad G(s) = \frac{s}{s^2+4}$

$\frac{s}{s^2(s^2+4)} = \boxed{\frac{1}{s(s^2+4)}}$

5. Use Laplace transforms to solve.
   a. $y'' + 3y' + 2y = u_2(t), y(0) = 0, y'(0) = 1$

$s^2F(s) - s(0) - 1 + 3(sF(s) - 0) + 2F(s) = \frac{e^{-2s}}{s}$

$F(s)(s^2+3s+2) - \cancel{1} = \frac{e^{-2s}}{s} + \frac{s}{s}$

$F(s) = \left(\frac{e^{-2s}+s}{s(s^2+3s+2)}\right) = \frac{e^{-2s}}{s(s+2)(s-1)} + \frac{\cancel{s}\,1}{\cancel{s}\,(s+2)(s-1)}$

$(s+2)(s-1)$

$\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$ $\quad A+B+C=0 \quad 3A-B+2C=0 \quad 2A=1$

$A(s^2+3s+2) + B(s^2-s) + C(s^2+2s)$ $\quad A = \frac{1}{2} \quad B = \frac{1}{6} \quad C = -\frac{2}{3}$

$\frac{A}{s+2} + \frac{B}{s-1}$ $\quad A+B=0 \quad -A+2B=1 \quad A = -\frac{1}{3} \quad B = \frac{1}{3}$

$As - A + Bs + 2B = 1$

$\longrightarrow$

   b. $y'' + 2y' + 2y = \delta(t-\pi), y(0) = 1, y'(0) = 0$

$s^2F(s) - s + 2(sF(s) - 1) + 2F(s) = e^{-\pi s}$

$F(s)(s^2+2s+2) - s - 2 = e^{-\pi s}$

$F(s)(s^2+2s+2) = e^{-\pi s} + s + 2$

$F(s) = \frac{e^{-\pi s} + s + 2}{s^2+2s+2} = \frac{e^{-\pi s} + (s+1) + 1}{(s+1)^2+1} = \frac{e^{-\pi s}}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$

$y(t) = e^{-t}\cos t + e^{-t}\sin t + u_\pi(t)\sin(t-\pi)e^{-t+\pi}$

5 cont'd

$$e^{-2s}\left(\frac{1/2}{s} + \frac{1/6}{s+2} + \frac{-2/3}{s-1}\right) + -\frac{1}{3}\left(\frac{1}{s+2}\right) + \frac{1}{3}\left(\frac{1}{s-1}\right)$$

$$y(t) = \frac{1}{2}u_2 + \frac{1}{6}e^{-2t}u_2 - \frac{2}{3}u_2e^{t} - \frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$$

$$\left(\frac{1}{2} + \frac{1}{e}e^{-2t} - \frac{2}{3}e^{t}\right)u_2 - \frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$$

Laplace transforms – Table

$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\dfrac{a}{s} \quad s > 0$	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$at \quad t \geq 0$	$\dfrac{a}{s^2}$	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
e^{-at}	$\dfrac{1}{s+a}$	$\sin(\omega t+\theta)$	$\dfrac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
te^{-at}	$\dfrac{1}{(s+a)^2}$	$\cos(\omega t+\theta)$	$\dfrac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$\dfrac{1}{2}t^2e^{-at}$	$\dfrac{1}{(s+a)^3}$	$t\sin\omega t$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$
$\dfrac{1}{(n-1)!}t^{n-1}e^{-at}$	$\dfrac{1}{(s+a)^n}$	$t\cos\omega t$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
e^{at}	$\dfrac{1}{s-a} \quad s > a$	$\sinh\omega t$	$\dfrac{\omega}{s^2-\omega^2} \quad s > \lvert\omega\rvert$
te^{at}	$\dfrac{1}{(s-a)^2}$	$\cosh\omega t$	$\dfrac{s}{s^2-\omega^2} \quad s > \lvert\omega\rvert$
$\dfrac{1}{b-a}(e^{-at}-e^{-bt})$	$\dfrac{1}{(s+a)(s+b)}$	$e^{-at}\sin\omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$\dfrac{1}{a^2}[1-e^{-at}(1+at)]$	$\dfrac{1}{s(s+a)^2}$	$e^{-at}\cos\omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$
t^n	$\dfrac{n!}{s^{n+1}} \quad n = 1,2,3..$	$e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
t^ne^{at}	$\dfrac{n!}{(s-a)^{n+1}} \quad s > a$	$e^{at}\cos\omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
t^ne^{-at}	$\dfrac{n!}{(s+a)^{n+1}} \quad s > a$	$1-e^{-at}$	$\dfrac{a}{s(s+a)}$
$\sqrt{t}$	$\dfrac{\sqrt{\pi}}{2s^{3/2}}$	$\dfrac{1}{a^2}(at-1+e^{-at})$	$\dfrac{1}{s^2(s+a)}$
$\dfrac{1}{\sqrt{t}}$	$\sqrt{\dfrac{\pi}{s}} \quad s > 0$	$f(t-t_1)$	$e^{-t_1 s}F(s)$
$g(t)\cdot p(t)$	$G(s)\cdot P(s)$	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$\int f(t)dt$	$\dfrac{F(s)}{s}+\dfrac{f^{-1}(0)}{s}$	$\delta(t)$ *unit impulse*	$1 \quad$ *all s*
$\dfrac{df}{dt}$	$sF(s)-f(0)$	$\dfrac{d^2f}{df^2}$	$s^2F(s)-sf(0)-f'(0)$
$\dfrac{d^nf}{dt^n}$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\cdots-f^{n-1}(0)$		

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$	2.	$\mathbf{e}^{at}$	$\dfrac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\dfrac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\dfrac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\dfrac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$	8.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\dfrac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\dfrac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\dfrac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\dfrac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\dfrac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\dfrac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\dfrac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			