MAT 212, Quiz #9, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Use a series to solve $y'' - xy' - y = 0$ centered at $x_0 = 0$.

$y=\sum_{n=0}^{\infty} a_n x^n \quad y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$

$y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$

$2(1)a_2(1) - a_0(1) + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} a_n x^n = 0$

$\cancel{2}a_2 = \frac{a_0}{2} \quad + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - n a_n - a_n\right]x^n = 0$

$(n+2)\cancel{(n+1)}a_{n+2} = \cancel{(n+1)}(a_n) \qquad a_{n+2} = \frac{a_n}{n+2}$

$a_0 = a_0$

$a_1 = a_1$

$a_2 = \frac{1}{2}a_0$

$a_3 = \frac{1}{3}a_1$

$a_4 = \frac{1}{4}\cdot\frac{1}{2}a_0 = \frac{1}{8}a_0$

$a_5 = \frac{1}{5}\cdot\frac{1}{3}a_1 = \frac{1}{15}a_1$

$a_6 = \frac{1}{6}\cdot\frac{1}{8}a_0 = \frac{1}{48}a_0$

$a_7 = \frac{1}{7}\cdot\frac{1}{15}a_1 = \frac{1}{105}a_1$

$y(x) = a_0\left(1+\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{48}x^6+\cdots\right)+a_1\left(x+\frac{1}{3}x^3+\frac{1}{15}x^5+\frac{1}{105}x^7+\cdots\right)$

2. Identify and classify the singular points of the equations.

a. $xy'' + (1-x)y' + xy = 0$

$y'' + \left(\frac{1-x}{x}\right)y' + y = 0$

Singular point at $x=0$ regular

b. $x(1-x^2)^3y'' + (1-x^2)^2y' + 2(1+x)y = 0$

$y'' + \frac{(1-x^2)^2}{x(1-x^2)^3}y' + \frac{2(1+x)}{x(1-x^2)^3}y = 0$

$y'' + \frac{1}{x(1-x)(1+x)}y' + \frac{2}{x(1-x)^3(1+x)^2}y = 0$

regular singular pt at $x=0$

regular singular pt at $x=-1$

irregular singular pt at $x=1$

3. Find $\vec{x}\cdot\vec{y}$ for $\vec{x} = \begin{pmatrix}1\\2\\4\end{pmatrix}, \vec{y} = \begin{pmatrix}-5\\2\\3\end{pmatrix}$.

$-5+4+12 = \boxed{11}$

4. Determine if the vectors $\begin{pmatrix}-3\\2\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}$ are linearly independent.

linearly independent
since det of matrix of vectors is $\neq 0$

(or row reduce)

5. Use series solutions to find a solution for $2xy'' + y' + xy = 0$ (Note: $x = 0$ is a regular singular point.)

$x=1$

$$[2(x-1)+2]y'' + y' + [(x-1)+1]y = 0$$

$$2(x-1)\sum_{n=2}^{\infty}(n-1)n a_n (x-1)^{n-2} + 2\sum_{n=2}^{\infty}(n-1)n a_n (x-1)^{n-2}$$
$$+ \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + (x-1)\sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$2\sum_{n=2}^{\infty}(n-1)n a_n (x-1)^{n-1} + 2\sum_{n=2}^{\infty}(n-1)n a_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^{n+1} + \sum_{n=-1}^{\infty}(n+3)(n+2)a_{n+3}(x-1)^{n+1} + \sum_{n=-1}^{\infty}(n+2)a_{n+2}(x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+1} + \sum_{n=-1}^{\infty} a_{n+1}(x-1)^{n+1} = 0$$

$n=-1$

$$2(1)a_2(1) + (1)a_1(1) + a_0(1) + \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n+3)(n+2)a_{n+3} + (n+2)a_{n+2} + a_n + a_{n+1}\right](x-1)^{n+1}$$

$2a_2 = -a_0 - a_1$

$a_2 = -\frac{1}{2}a_0 - \frac{1}{2}a_1$

$$(n+2)(n+1)a_{n+2} + (n+3)(n+2)a_{n+3} + (n+2)a_{n+2} + a_n + a_{n+1} = 0$$

$$(n+2)(n+2)a_{n+2} + (n+3)(n+2)a_{n+3} + a_n + a_{n+1} = 0$$

$$a_{n+3} = -a_n - a_{n+1} - (n+2)^2 a_{n+2}$$

$n=0$ $a_3 = -a_0 - a_1 - 4a_2$
$\left(-\frac{1}{2}a_0 - \frac{1}{2}a_1\right) = a_0 + a_1$

$n=1$ $a_4 = -a_1 - a_2 - 9a_3$

$-a_1 + \frac{1}{2}a_0 + \frac{1}{2}a_1 - 9a_0 - 9a_1$

$= -\frac{17}{2}a_0 - \frac{19}{2}a_1$

$n=2$

$a_5 = -a_2 - a_3 - 16a_4$

$\left(\frac{1}{2}a_0 + \frac{1}{2}a_1\right) - (a_0 + a_1) - 16\left(-\frac{17}{2}a_0 - \frac{19}{2}a_1\right)$

$\frac{1}{2}a_0 - a_0 + 136a_0 + \frac{1}{2}a_1 - a_1 + 152a_1$

$\frac{271}{2}a_0 + \frac{303}{2}a_1$

$$y(x) = a_0\left(1 - \frac{1}{2}(x-1)^2 + (x-1)^3 - \frac{17}{2}(x-1)^4 + \frac{271}{2}(x-1)^5 + \dots\right) + a_1\left(x - \frac{1}{2}(x-1)^2 + (x-1)^3 - \frac{19}{2}(x-1)^4 + \frac{303}{2}(x-1)^5 + \dots\right)$$