MTH 174, Exam #3, Spring 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Find the limit of the **sequence** $a_n = \cos\frac{2}{n}$. (4 points)

$$\lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = \cos(0) = 1$$

2. Simplify $\frac{(2n+2)!}{(2n)!}$. (3 points)

$$\frac{(2n+2)(2n+1)\cancel{(2n)}\cancel{(2n-1)}\cdots\cancel{(3)}\cancel{(2)}\cancel{(1)}}{\cancel{(2n)}\cancel{(2n-1)}\cdots\cancel{(3)}\cancel{(2)}\cancel{(1)}} = (2n+2)(2n+1)$$

3. Determine if the **sequence** $a_n = \frac{3n}{n+2}$ is bounded and monotonic. (6 points)

$a_n \geq 0$ since $n \geq 0$ bounded below

$$f(x) = \frac{3x}{x+2}$$

$$f'(x) = \frac{3(x+2)-(1)(3x)}{(x+2)^2} = \frac{\cancel{3x}+6-\cancel{3x}}{(x+2)^2} = \frac{6}{(x+2)^2} \geq 0 \text{ for all } x$$

always positive monotonic

$$3 \overset{?}{\geq} \frac{3n}{n+2} \Rightarrow 3n+6 \overset{?}{\geq} 3n \Rightarrow 6 \geq 0 \text{ true}$$

So a_n bounded above by 3

bounded and monotonic

4. Find the sum of series. (6 points each)

a. $\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$

$a = 5$

$r = \frac{2}{3}$

$$\frac{5}{1-\frac{2}{3}} = \frac{5}{\frac{1}{3}} = \boxed{15}$$

b. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$ =

$$\frac{A}{n} + \frac{B}{n+2} = \frac{An+2A+Bn = 4}{n(n+2)}$$

$2A = 4 \Rightarrow A = 2$

$A + B = 0 \Rightarrow B = -2$

$$\sum_{n=1}^{\infty} \frac{2}{n} - \frac{2}{n+2}$$

$$\frac{2}{1} + \frac{2}{2} - \lim_{n\to\infty}\left(\cancel{\frac{2}{n+1} + \frac{2}{n+2}}^{\,0}\right) = 2+1 = \boxed{3}$$

5. Use the integral test to determine if the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges or diverges. (6 points)

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx \qquad u = \ln x \quad dv = \frac{1}{x^2}dx \qquad du = \frac{1}{x} \quad v = \frac{-1}{x}$$

$$-\frac{\ln x}{x} + \int \frac{1}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x}\Big|_1^{\infty} = \lim_{x\to\infty}\left(\frac{-\ln x}{x} - \cancel{\frac{1}{x}}^{\,0}\right) + 0 + 1$$

$$= 1 + \lim_{x\to\infty} \cancel{\frac{-\frac{1}{x}}{1}}^{\to 0} = 1 \qquad \text{Converges}$$

6. Find the value of N needed to approximate the sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ within 0.001 of the true value. (7 points)

$$\int_N^\infty \frac{1}{x^{3/2}}\,dx = \int_N^\infty x^{-3/2}\,dx = -2x^{-1/2}\Big|_N^\infty = \frac{-2}{\sqrt{x}}\Big|_N^\infty = 0 + \frac{2}{\sqrt{N}} \le .001$$

$2000 \approx \sqrt{N}$

$\boxed{N = 4{,}000{,}000}$

7. Determine if the series converges. Clearly state the test used. (4 points each)

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

limit Comparison w/ $\frac{1}{n^{3/2}}$ (converges by p-series or question #6)

$$\lim_{n\to\infty} \frac{\frac{1}{n^{3/2}}}{\frac{1}{\sqrt{n^3+1}}} = \lim_{n\to\infty} \frac{\sqrt{n^3+1}}{n^{3/2}} = 1$$

converges

b. $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$

$$\lim_{n\to\infty} \left| \frac{n+1}{(n+2)\,2^n} \cdot \frac{(n+1)2^{n-1}}{n} \right| = \lim_{n\to\infty} \left(\frac{(n+1)\overset{1}{\cancel{(n+1)}}}{\cancel{n}(n+2)} \right) \left(\frac{2^{n-1}}{2^n} \right) = \lim_{n\to\infty} \left(\frac{1}{2} \right) = \frac{1}{2}$$

Converges by ratio test

c. $\sum_{n=2}^{\infty} \frac{1}{5^n+1}$

limit comparison w/ $\frac{1}{5^n}$ (converges by geometric series test)

$$\lim_{n\to\infty} \frac{\frac{1}{5^n+1}}{\frac{1}{5^n}} = \lim_{n\to\infty} \frac{5^n}{5^n+1} = 1$$

converges

d. $\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^n}$

$\lim_{n\to\infty} \sqrt[n]{\frac{6^n}{(n+1)^n}} = \lim_{n\to\infty} \frac{6}{n+1} = 0$ Converges by root test

e. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$

$\lim_{n\to\infty} \frac{\sqrt{n}}{n+2} = 0$ converges by alternating series test

f. $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$ (by e^x Taylor series)

$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0$ Converges by ratio test

g. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

$\lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{(\ln n)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\ln n} = \frac{\lim_{n\to\infty} \sqrt[n]{n}}{\lim_{n\to\infty} \ln n} = \frac{1}{\lim_{n\to\infty} \ln n}$

$= 0$

Converges by root test

8. Determine if the series $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$ converges conditionally or absolutely. (6 points)

$\cos n\pi = (-1)^n \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

$\lim_{n\to\infty} \frac{1}{n+1} = 0$ converges if alternating

Converges conditionally

$\lim_{n\to\infty} \frac{\frac{1}{n}}{\left|\frac{1}{n+1}\right|} = \lim_{n\to\infty} \frac{n+1}{n} = 1$ diverges if positive

(by limit comp. w/ $\frac{1}{n}$)

9. Determine the number of terms needed to estimate the value of the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ within 0.001 of the true value. (7 points)

$\frac{1}{n^5} \leq .001$

$n^5 \geq 1000$

$n \geq 3.98\ldots$

$\boxed{n=4}$

10. Determine the interval and radius of convergence of x for the series below. Be sure to check the endpoints. (5 points each)

a. $\sum_{n=0}^{\infty} \frac{(-1)^n(x+1)^n}{n}$

$\lim_{n\to\infty} \left|\frac{(x+1)^{n+1}}{n+1} \cdot \frac{n}{(x+1)^n}\right| = |x+1| < 1$

$\boxed{\begin{array}{c} R=1 \\ (-2,0] \end{array}}$

$-1 < x+1 < 1$
$\quad -1 \quad\; -1 \quad -1$

$-2 < x < 0$

if $x=-2$
$\sum \frac{(-1)^n(-1)^n}{n}$ diverges p-series

if $x=0$
$\sum \frac{(-1)^n(1)^n}{n}$ converges alt.

b. $\sum_{n=0}^{\infty} n!\left(\frac{x}{2}\right)^n$

$\lim_{n\to\infty} \left|\frac{(n+1)!\left(\frac{x}{2}\right)^{n+1}}{n!\left(\frac{x}{2}\right)^n}\right| = \lim_{n\to\infty} \left|(n+1)\left(\frac{x}{2}\right)\right|$

converges only if $x=0$

$\boxed{\begin{array}{c} R=0 \\ \{0\} \end{array}}$

11. Write the functions as a power series centered at c. (6 points each)

a. $f(x) = \frac{4}{3x+2}, c = 3$

$$\frac{4}{3(x-3)+9+2} = \frac{4}{11+3(x-3)} \cdot \frac{\frac{1}{11}}{\frac{1}{11}} = \frac{4/11}{1+\frac{3}{11}(x-3)}$$

$a = 4/11, \quad r = -\frac{3}{11}(x-3)$

$$= \sum_{n=0}^{\infty} \frac{4}{11}\left[-\frac{3}{11}(x-3)\right]^n = \sum_{n=0}^{\infty} \frac{4}{11}\left(\frac{-3}{11}\right)^n (x-3)^n = \sum_{n=0} \frac{4(-3)^n}{(11)^{n+1}}(x-3)^n$$

b. $f(x) = \frac{4x}{(x+1)^3}$

$$\left(\sum_{n=0}^{\infty} ar^n\right)'' = \left(\sum_{n=1}^{\infty} anr^{n-1}\right)' = \sum_{n=2} an(n-1)r^{n-2}$$

$a = 2x$

$r = -x$

$$(a(1-r)^{-1})'' = (a(1-r)^{-2})' = 2a(1-r)^{-3} = \frac{2a}{(1-r)^3}$$

$$\sum_{n=2}^{\infty} 2xn(n-1)(-x)^{n-2} = \sum_{n=2}^{\infty} 2n(n-1)(-1)^n x^{n-1}$$

12. Find a Taylor polynomial for the function centered at c for the given number of terms. (6 points each)

a. $f(x) = \ln(x), n = 4, c = 2$

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(2)$	$(x-2)^n$	$\frac{f^{(n)}(c)(x-c)^n}{n!}$
0	1	$\ln x$	$\ln 2$	1	$\ln 2$
1	1	$\frac{1}{x}$	$\frac{1}{2}$	$x-2$	$\frac{1}{2}(x-2)$
2	2	$\frac{-1}{x^2}$	$\frac{-1}{4}$	$(x-2)^2$	$-\frac{1}{8}(x-2)^2$
3	6	$\frac{2}{x^3}$	$\frac{1}{4}$	$(x-2)^3$	$\frac{1}{24}(x-2)^3$
4	24	$\frac{-6}{x^4}$	$\frac{-3}{8}$	$(x-2)^4$	$-\frac{1}{64}(x-2)^4$

$$P_4 = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4$$

b. $f(x) = xe^{x/3}, n = 4, c = 0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{x/3} = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{3}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{3^n n!}$$

$$xe^{x/3} = \sum_{n=0}^{\infty} x \cdot \frac{x^n}{3^n n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{3^n n!}$$

$$P_4 = \frac{x}{3^0 \cdot 1} + \frac{x^2}{3 \cdot 1} + \frac{x^3}{9 \cdot 2} + \frac{x^4}{27 \cdot 6}$$

$$= x + \frac{x^2}{3} + \frac{x^3}{18} + \frac{x^4}{162}$$

13. Use Taylor and power series (see the table at the end of the test) to find a power series for expression. (6 points each)

a. $f(x) = \frac{e^x - 1}{x}$

$$\frac{-1 + e^x}{x} = \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots - 1}{x} = \frac{x + \frac{x^2}{2} + \frac{x^3}{3} + \dots}{x} = 1 + \frac{x}{2} + \frac{x^2}{3} + \dots$$

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

b. $f(x) = \int \cos x^2 \, dx$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

$$\int \cos x^2 \, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!\,(4n+1)}$$

14. Find the maximum error on the Taylor polynomial with $n = 6$ for the function $f(x) = e^{-x}$, centered at $c = 0$, at $x = 0.5$. The remainder formula is $R_n \leq \frac{(x-c)^{n+1}}{(n+1)!} \max |f^{n+1}(z)|$. (7 points)

$f(x) = e^{-x}$, $f'(x) = -e^{-x}$, $f''(x) = e^{-x}$, $f'''(x) = -e^{-x}$, $f^{IV}(x) = e^{-x}$,
$f^{V}(x) = -e^{-x}$, $f^{VI}(x) = e^{-x}$, $f^{VII}(x) = -e^{-x}$ max $|-e^{-x}|$ at 0

$$R_n \leq \frac{(.5-0)^7 |1|}{7!} = \frac{(.5)^7}{7!} = \frac{1/128}{5040} = \boxed{1.55 \times 10^{-6}}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots \qquad R = 1$$