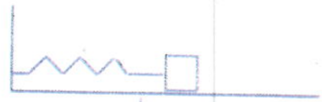
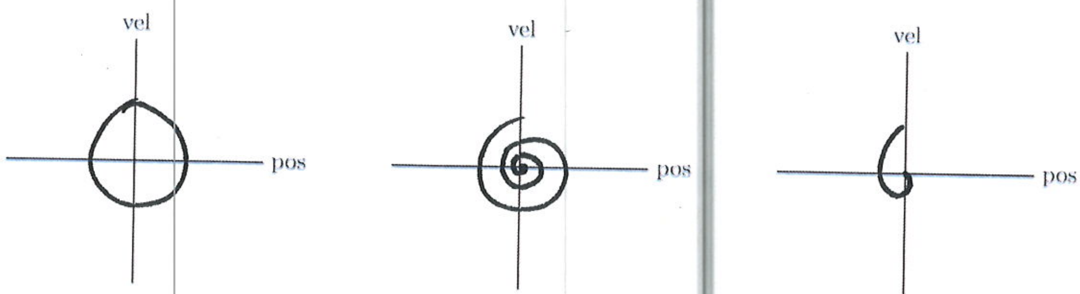
Spring-Mass Motion Investigation

In this problem we use Newton's Law of motion ($\sum F = ma$) to develop a system of rate of change equations in order to be able to describe, explain, and predict the motion of a mass attached to a spring.



$x < 0$ $\quad x = 0$ $\quad x > 0$

1. Depending on the values for parameters like the stiffness of the spring k, the weight of the object attached to the spring m, and the amount of friction, different behaviors may be possible. Imagine for a set spring and mass you vary the amount of friction on the surface. What do you imagine the various position versus velocity graphs would look like? Provide rough sketches.



Hint: think of a real spring and plot key points of oscillation

etc.

← what is velocity here?

2. Use Newton's Law of motion to develop a rate of change equation to model the motion of an object on a spring. Assume that the only forces acting on the object are the spring force ($-kx$, where k is the spring constant) and the friction force (assumed to be proportional to the velocity, namely $-b\frac{dx}{dt}$, where b is the damping coefficient).

$$ma = -bv - kx$$

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

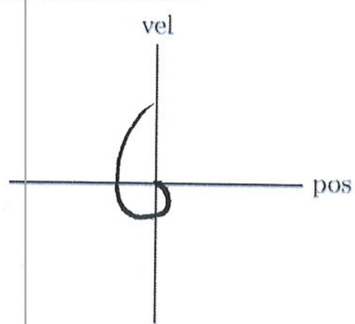
3. Application of Newton's Law of Motion to the spring-mass situation in the previous problem results in the following:

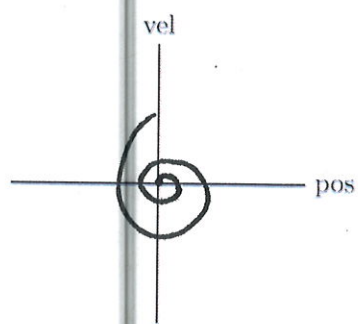
$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0,$$

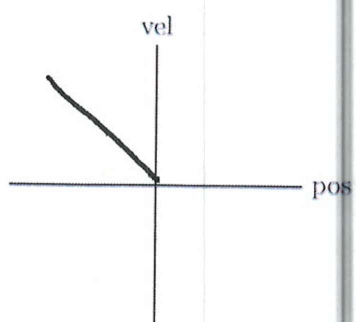
where x is the position of the object attached to the end of the spring, m is the mass of the object, b is the friction parameter (also called damping coefficient), and k is the spring constant. Because $\frac{dx}{dt} = y$, where y is the velocity, and $\frac{dy}{dt} = \frac{d^2x}{dt^2}$, we can converting this to a system of two differential equations as follows:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\frac{k}{m}x - \frac{b}{m}y$$

Use the GeoGebra applet https://ggbm.at/vT5tgWrg to investigate the motion of the object as depicted in the phase plane when $m = 1$, the spring constant $k = 2$, and the friction parameter, b, varies between 0 and 4. In particular, how does the vector field (and corresponding behavior of the mass) change when the friction parameter increases from 0 to say 2, 2.3, 3, or 3.8? Use the space below to record your observations.



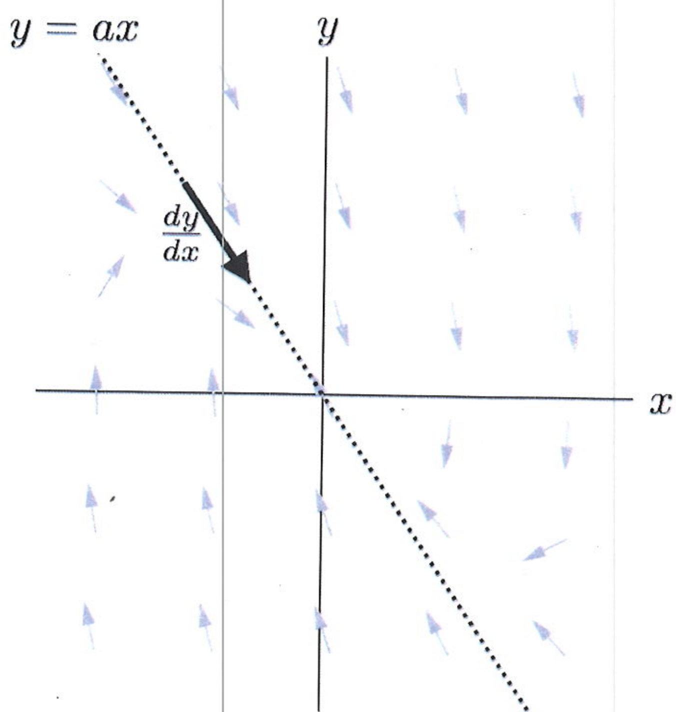




4. Joey and Kara set the friction parameter to 3, resulting in the following system:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - 3y$$

They notice that graphs of solutions in the position-velocity plane seem to get pulled into the origin along a straight line. Help Joey and Kara figure out how to use algebra to find the slope of this straight line.



$y = ax$

$\frac{dy}{dt} = a\frac{dx}{dt}$

$\frac{dy}{dt} = ay$

$\frac{dy}{dt} = a(ax)$

$\frac{dy}{dt} = -2x - 3y$

$\frac{dy}{dt} = -2x - 3ax$

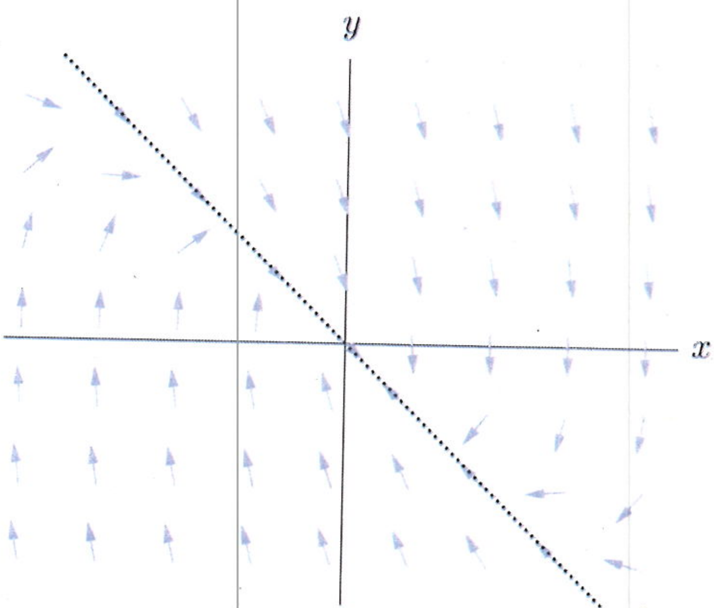
$a^2x + 3ax + 2x = 0$

$x(a^2 + 3a + 2) = 0$

$(a+2)(a+1) = 0$

$a = -2, a = -1$

5. Continuing their investigation with the friction parameter set to 3, Kara and Joey are working to find the slope of the observed straight line. Joey sets up the equation $\frac{-2x-3y}{y} = \frac{y}{x}$ and Kara sets up the equation $\frac{-2x-3ax}{ax} = a$. Interpret Joey's and Kara's equations and then solve both.



$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2x-3ax}{ax}$$

$$\frac{y}{0x} = \frac{ax}{x} = a \quad \text{Slope of line}$$

$$\frac{-2x-3ax}{ax} = a$$

$$-2x-3ax = ax^2$$

$$a^2x + 3ax + 2x = 0$$

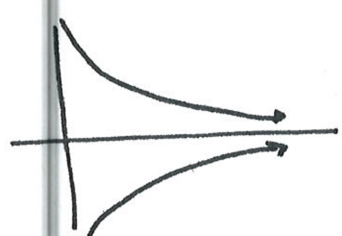
$$x(a^2+3a+2) = 0$$

$$(a+2)(a+1) = 0$$

$$a = -2, \quad a = -1$$

6. Place your finger on the dotted line starting in the second quadrant and trace out the path that the mass takes, as represented in the phase plane. Describe what happens to your finger and relate this to the motion of the mass.

answers will vary
as position gets closer to origin,
velocity also gets smaller



7. Joey found another straight line solution when the friction parameter b was set to 1. Use algebra to find the slope or explain why he is mistaken.

$$\frac{-2x-ax}{ax} = a$$

$$a^2x + ax + 2x = 0$$

$$x(a^2+a+2) = 0$$

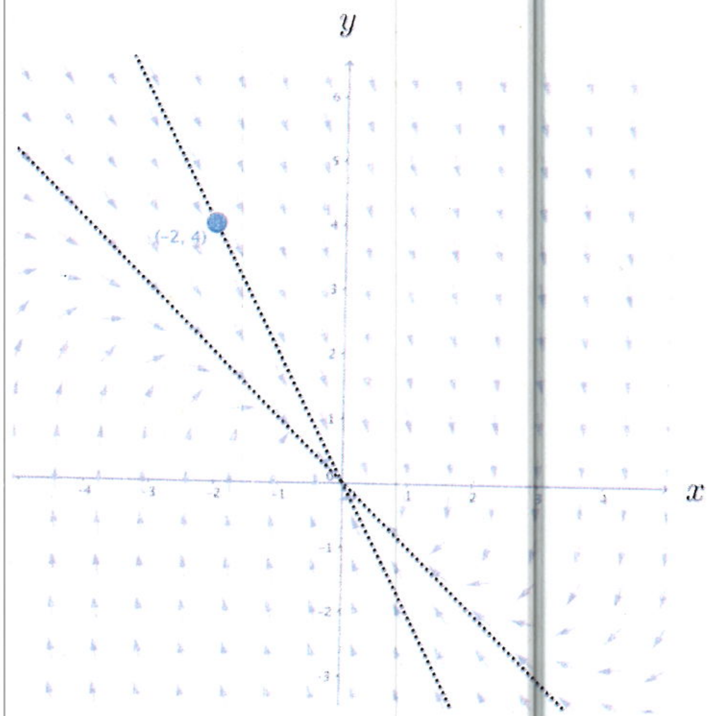
$$\frac{-1 \pm \sqrt{1-8}}{2} = a$$

Solution is complex
direction field does not have
straight line in slope field

8. In your investigation of the spring-mass system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - 3y$$

you should have found that when the friction parameter was equal to 3, solutions with initial conditions that are either on the line $y = -x$ or on the line $y = -2x$ head directly toward the origin along a straight path.



For the initial condition (-2, 4), what are the equations for $x(t)$ and $y(t)$? Hint: substitute $y = -2x$ and $x = -y/2$ into dx/dt and dy/dt, respectively.

$\frac{dy}{dt} = ay = -2y$

$\int \frac{dy}{y} = \int -2\,dt$

$\ln y = -2t + C$

$y = Ae^{-2t}$

$y = 4e^{-2t}$

$\frac{dx}{dt} = -2x$

$\int \frac{dx}{x} = \int -2\,dt$

$\ln x = -2t + C$

$x = Be^{-2t}$

$x = -2e^{-2t}$

9. Susan notices that the $x(t)$ and $y(t)$ equations have the same exponent, and then makes the conjecture that along any straight line solution, $x(t)$ and $y(t)$ **must** have the same exponent. Do you agree with her conjecture? Why or why not?

yes, since y is $\frac{dx}{dt}$ and $\frac{d}{dt}(-2e^{-2t}) = 4e^{-2t}$

exponential always stays same exponential

10. (a) What are the $x(t)$ and $y(t)$ equations for the solution with initial condition (-1, 2)? What does the 3D graph of this solution look like?

$x = -e^{-2t}$

$y = 2e^{-2t}$

exponential decay in x, t, or y, t

straight line in x, y

(b) If you multiplied $x(t)$ and $y(t)$ equations from problem 10a by some number, say -3 for example, is the result also a solution to the system of differential equations? Algebraically show that your conclusion is correct.

$x = 3e^{-2t}$

$y = -6e^{-2t}$

$\frac{dx}{dt} = y = -6e^{-2t} = \frac{d}{dt}(3e^{-2t}) = -6e^{-2t}$

$\frac{dy}{dt} = -2(3e^{-2t}) - 3(-6e^{-2t}) = -6e^{-2t} + 18e^{-2t}$

$= 12e^{-2t} = \frac{d}{dt}(-6e^{-2t}) = 12e^{-2t}$

Satisfies equations

(c) What are the $x(t)$ and $y(t)$ equations for *any* solution with initial condition along the line $y = -2x$?

$(x_0, y_0) \rightarrow$ $x = x_0 e^{-2t}$

$y = -2x_0 e^{-2t}$

$-2x_0 = y_0$

11. For the initial condition (-2, 2), what are the equations for $x(t)$ and $y(t)$? What are the $x(t)$ and $y(t)$ equations for *any* solution with initial condition along the line $y = -x$?

$x = -2e^{-t}$

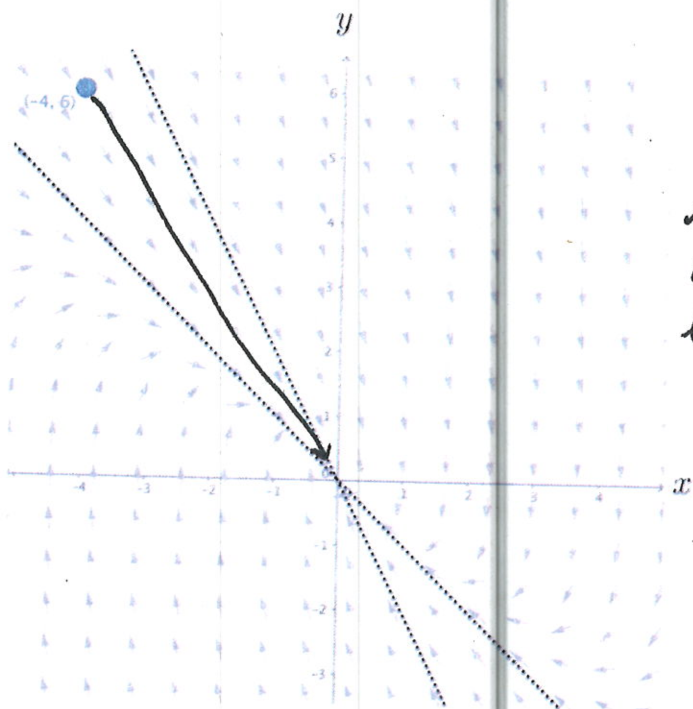
$y = 2e^{-t}$

$\frac{dx}{dt} = ax$

$x = Ce^{ax}$

$a = -1$

12. (a) Suppose you were to start with an initial condition somewhere in the second quadrant between the two straight line solutions, say at (-4, 6). Sketch what you think the solution as viewed in the phase plane looks like and explain your reasoning.



it collapses to origin although not necessarily on a perfectly straight line

(b) Notice that (-4, 6) is a linear combination of the initial conditions (-2, 4) and (-2,2). Show that the solution with the initial condition (-4, 6) is also a linear combination of the solutions with initial conditions (-2, 4) and (-2,2).

$(-2,4) + (-2,2) = (-4,6)$ vector addition

$x = -2e^{-2t} + -2e^{-t}$

$y = 4e^{-2t} + 2e^{-t}$

(c) According to your result in 12b, what does the solution in the phase plane look? Explain your reasoning.

tends to drift slightly toward $y=-x$ since it will survive longer

13. (a) What are the $x(t)$ and $y(t)$ equations are for the solution with initial condition $(2, 5)$?

$$a(-2,4) + b(-2,2) = (2,5)$$

$$-2a - 2b = 2 \qquad a = \frac{7}{2}$$
$$4a + 2b = 5 \qquad b = -\frac{9}{2}$$

$$x = \frac{7}{2}e^{-2t} - \frac{9}{2}e^{-t}$$

(b) **According to your $x(t)$ and $y(t)$ equations**, what does the solution in the phase plane look like? Explain your reasoning and provide a sketch. Use the GeoGebra applet, https://ggbm.at/cMSUC7qR to corroborate your conclusion.

answers will vary, see graph

(c) Develop an argument that almost all graphs of solutions in the phase plane head into the origin with a slope of -1.

$e^{-2t} \to 0$ faster than e^{-t}

so over time the term w/ e^{-t} will dominate solution

14. Figure out a general approach for determining the $x(t)$ and $y(t)$ equations for any initial condition.

$$x = Ae^{-2t} + Be^{-t}$$
$$y = -2Ae^{-2t} - Be^{-t}$$

where (x_0, y_0) satisfies at $t=0$

$$A + B = x_0$$
$$-2A - B = y_0$$