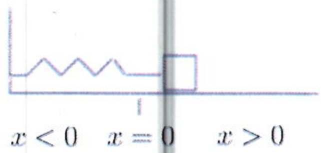
Spiraling Solutions - Spring Mass Revisited

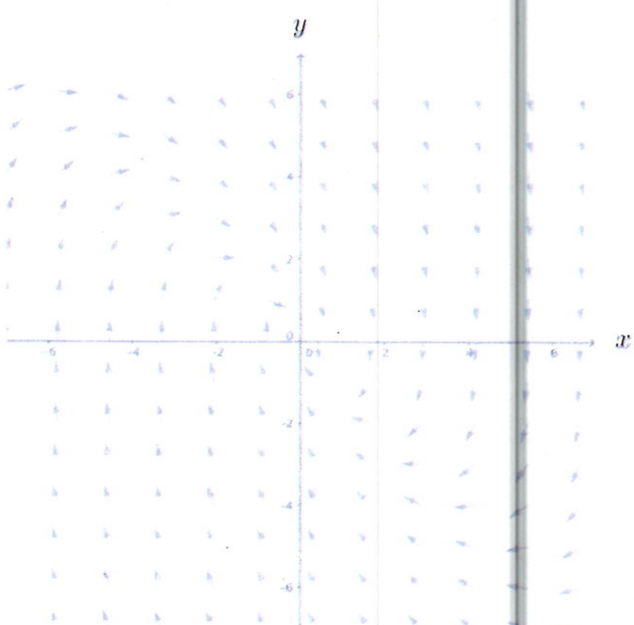
In a previous problem we applied Newton's law of motion for a spring mass system and obtained the second order differential equation $\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$, where x is the position of the object attached to the end of the spring, m is the mass of the object, b is the damping coefficient, and k is the spring constant. Using the fact that velocity is the derivative of position and choosing the mass $m = 1$ and the spring constant $k = 2$, we converted this to the following system of two differential equations:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - by$$



We were able to figure out the $x(t)$ and $y(t)$ equations when the value of the friction parameter was such that there were straight line solutions in the phase plane. Such a situation is typically referred to as *overdamped*. The situation is called *damped* when the differential equations predict that the mass will oscillate about the 0 position and *undamped* when there is no friction. In the following problems we figure out the $x(t)$ and $y(t)$ equations for the damped. We consider the undamped situation in the homework.

The vector field for the case when $b = 2$ is shown below. Based on this vector field, it appears that the differential equations predict that the mass will oscillate back and forth. Even though there are not any straight line solutions, we can still use the same algebraic approach as before to get the $x(t)$ and $y(t)$ equations for any initial condition, but we will have to deal complex numbers. Problems 1-7 outline a way to do this.



1. For the system of differential equations

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - 2y$$

use the same algebraic approach as before to verify that the slopes of the "straight line" solutions are $-1 \pm i$.

$y = ax \longrightarrow \frac{dy}{dx} = a$

$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2x-2y}{y} = \frac{-2x-2ax}{ax} = a$

$a^2x + 2ax + 2x = 0$

$x(a^2 + 2a + 2) = 0$

$a = \frac{-2 \pm \sqrt{4-8}}{2} =$

$a = \frac{-2 \pm 2i}{2} = -1 \pm i$

2. For solutions with "straight line" slope $y = (-1+i)x$, find the $x(t)$ and $y(t)$ equations (in terms of complex numbers) for the solution along this "straight line" with initial condition $(1, -1+i)$.

$\frac{dx}{dt} = ax$

$\int \frac{dx}{x} = \int a\,dt$

$\ln x = Ae^{at}$

$x = e^{(-1+i)t}$

$y = (-1+i)e^{(-1+i)t}$

3. For solutions with "straight line" slope $y = (-1-i)x$, find the $x(t)$ and $y(t)$ equations (in terms of complex numbers) for the solution along this "straight line" with initial condition $(1, -1-i)$.

$x = e^{(-1-i)t}$

$y = (-1-i)e^{(-1-i)t}$

4. Use Euler's formula $e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b)$ to rewrite the $x(t)$ and $y(t)$ equations from problem 2 (call these $x_1(t)$ and $y_1(t)$) and then again from problem 3 (call these $x_2(t)$ and $y_2(t)$).

$$e^{(-1+i)t} = e^{-t} e^{it} = e^{-t}(\cos t + i \sin t)$$

$$e^{(-1-i)t} = e^{-t} e^{-it} = e^{-t}(\cos(-t) + i \sin(-t)) =$$
$$e^{-t}(\cos t - i \sin t)$$

5. Denise suggests that if you add $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ to $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ the resulting pair of equations is (i) real valued and (ii) a solution to the same system of differential equations. Verify that this is true.

$$e^{-t}(\cos t + i \sin t)$$
$$e^{-t}(\cos t - i \sin t)$$
$$\overline{}$$
$$e^{-t}(2\cos t)$$

6. Verify that if you subtract $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ from $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ and multiply the result by the complex number i, then the resulting pair of equations will be a real and a solution to the same system of differential equations.

$$e^{-t}(\cos t + i \sin t)$$
$$e^{-t}(-\cos t + i \sin t)$$
$$\overline{}$$
$$e^{-t}(2i \sin t) \quad \div i \quad \text{or } (\times i)$$

$$e^{-t}(2 \sin t) \qquad\qquad -e^{-t}(2 \sin t)$$

7. (a) Form the general solution to the system of differential equations

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - 2y$$

(b) What aspect of your general solution could be interpreted as the effect of friction on the spring mass system?

(c) Find the particular solution for the initial condition (2, 3) and sketch the x vs t and y vs t graphs.

e^{-t} causes oscillations to decay, so this could be friction

$X = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$ $\qquad \binom{2}{3} \leftarrow x' \qquad t=0$

$2 = c_1(1)(1) + \cancel{c_2(1)(0)} \qquad c_1 = 2$

$y = -c_1 e^{-t}\cos t + -c_1 e^{-t}\sin t + -c_2 e^{-t}\sin t + c_2 e^{-t}\cos t$

$3 = -2(1)(1) - \cancel{2(1)(0)} - \cancel{c_2(1)(0)} + c_2(1)(1)$

$3 = -2 + c_2$

$c_2 = 5$

$X = 2e^{-t}\cos t + 5e^{-t}\sin t$