Equilibrium Solutions for Linear Systems

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

1. For each part below, use two different ways (one algebraic and one geometric using nullclines) to figure out the number and location of equilibrium solutions.

   (a) $\frac{dx}{dt} = 3x + 2y$, $\frac{dy}{dt} = -2y$

   $0 = 3x + 2y$
   $\underline{0 = \quad -2y}$
   $0 = 3x \quad x = 0, y = 0$

   

   one solution (0,0)

   (b) $\frac{dx}{dt} = 4x - 2y$, $\frac{dy}{dt} = -2x + y$

   $4x - 2y = 0$
   $\underline{2(-2x + y = 0)}$
   $0 = 0$

   $y = 2x$

   

   infinite solutions $y = 2x$

2. Is it possible to find values of a, b, c, d such that the system of differential equations

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

has exactly two equilibrium solutions? Explain why or why not.

no. Systems of linear equation have only 3 possible solution types: 1) no solution, 2) one solution, 3) infinite solution. Option #1 is foreclosed if the system is homogeneous

3. Develop criteria (in terms of the parameters $a, b, c,$ and d) that tell us about the number and location of equilibrium solutions for systems of differential equations of the form

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

$ax + by = 0 \quad * c$
$cx + dy = 0 \quad * a$

$acx + bcy = 0$
$\underline{-acx - ady = 0}$
$(bc - ad)y = 0$

one solution when $bc - ad$ (or $ad - bc$) $\neq 0$

infinite solutions if $bc - ad$ or $(ad - bc)$ is equal to zero

Matrix Notation and Equilibrium Solutions for Linear Systems

$$\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy\end{aligned}$$

One way to approach problem 3 is to think about there being an infinite number of equilibrium solutions when the two nullclines coincide. That is, when the equations $\begin{aligned}0 &= ax + by \\ 0 &= cx + dy\end{aligned}$ determine the same set of points. Put another way, the equations are *dependent* when $y = -\frac{a}{b}x$ and $y = -\frac{c}{d}x$ are the same equation. Thus, $-\frac{a}{b} = -\frac{c}{d}$, which says that $-ad = -cb$. Rewriting this yields $ad - bc = 0$.
As shown next, another way to arrive at this result is to use matrix notation and the fact that two equations are dependent when the determinant of the matrix is zero.

$$\begin{aligned}ax + by &= 0 \\ cx + dy &= 0\end{aligned} \Longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus, the equations $\begin{aligned}ax + by &= 0 \\ cx + dy &= 0\end{aligned}$ are dependent when the determinant of the coefficient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is zero. That is, when $ad - bc = 0$.

Next, we develop an approach for finding the general solution to a system of differential equations of the form $\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy\end{aligned}$ by first finding the value of the exponent (that is, the **eigenvalue**) associated with any straight line solution *before* finding the slope of the straight line solutions (typically called **eigensolutions**). Note that in your previous work you first found the slope of straight line solutions and then found the exponent. Some students have referred to this as the "slope first" method. In the pages that follow, an alternative approach is developed the "eigenvalue first" method.

We develop this alternative method for four reasons:

- The eigenvalue first method can be used for systems of three or more differential equations whereas the slope first method cannot.
- Oftentimes just knowing the eigenvalues is sufficient for understanding the overall picture of solutions in the phase plane and so therefore this method is more efficient.
- The eigenvalue first approach makes important connections with linear algebra.
- The eigenvalue first approach is algebraically more efficient.

Eigenvalue First Method

For linear systems of the form $\begin{array}{rcl}\frac{dx}{dt} & = & ax+by \\ \frac{dy}{dt} & = & cx+dy\end{array}$, one way to determine the exponent (i.e. λ, the eigenvalue) for possible straight line solutions (or eigensolutions) is to use the fact that if eigensolutions exist in the phase plane, then $\frac{dx}{dt} = \lambda x$ and $\frac{dy}{dt} = \lambda y$.

4. Explain why this has to be true.

these are straight line solutions

Combining the fact that $\begin{array}{rcl}\frac{dx}{dt} & = & ax+by \\ \frac{dy}{dt} & = & cx+dy\end{array}$ with the fact that for straight line solutions $\frac{dx}{dt} = \lambda x$ and $\frac{dy}{dt} = \lambda y$ along the straight line, we can set up the following two equations:

$$\begin{array}{rl} ax+by & = \lambda x \\ cx+dy & = \lambda y \end{array} \tag{4}$$

Rearranging these equations we get

$$\begin{array}{rl} (a-\lambda)x+by & = 0 \\ cx+(d-\lambda)y & = 0 \end{array} \tag{5}$$

Note that although these equations look similar to the nullcline equations, the coefficients are different.

In order to get straight line solutions, with a particular value of λ corresponding to an exponent from the straight line solution, the equations from (5) need to be dependent.

5. Explain why this has to be true.

if the revised system in #4 (top) has a solution, then the bottom one must be dependent. if the bottom equation in (5) is a single solution situation, then x=0, y=0 is the only case and λ can be anything. for some λ to exist then (0,0) can't be the only solution.

Rewriting these dependent equations in slope form yields $y = -\frac{a-\lambda}{b}x$ and $y = -\frac{c}{d-\lambda}x$ and thus $-\frac{a-\lambda}{b} = -\frac{c}{d-\lambda}$. Rearranging this last equation we get the following:

$$
\begin{aligned}
&(a-\lambda)(d-\lambda) - bc = 0 \\
\Rightarrow &\lambda^2 - (a+d)\lambda + (ad-bc) = 0 \\
\Rightarrow &\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}
\end{aligned}
$$

We can more efficiently obtain this same result using matrix notation and the fact that two equations are dependent when the determinant of the coefficient matrix is zero as follows:

$$
\begin{aligned}
(a-\lambda) + by &= 0 \\
cx + (d-\lambda)y &= 0
\end{aligned}
\Longrightarrow
\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

Thus, the equations $\begin{aligned}(a-\lambda) + by &= 0 \\ cx + (d-\lambda)y &= 0\end{aligned}$ are dependent when the determinant of the coefficient matrix

$$
\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}
$$

is zero. That is, when $(a-\lambda)(d-\lambda) - bc = 0$.

EXAMPLE:

Determine the general solution for the system of differential equations

$$
\begin{aligned}
\frac{dx}{dt} &= 4x + 2y \\
\frac{dy}{dt} &= x + 3y
\end{aligned}
$$

using the "eigenvalue first" approach.

In order to get eigensolutions, we need to have

$$
\begin{aligned}
&\begin{aligned} 4x + 2y &= \lambda x \\ x + 3y &= \lambda y \end{aligned} && (6) \\
\Rightarrow &\begin{aligned} (4-\lambda)x + 2y &= 0 \\ x + (3-\lambda)y &= 0 \end{aligned} && (7) \\
\Rightarrow &\begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (8) \\
\Rightarrow &(4-\lambda)(3-\lambda) - 2 = 0 && (9) \\
\Rightarrow &\lambda^2 - 7\lambda + 10 = 0 && (10) \\
\Rightarrow &(\lambda - 5)(\lambda - 2) = 0 && (11) \\
\Rightarrow &\lambda = 2, \lambda = 5 && (12)
\end{aligned}
$$

<u>For $\lambda = 2$</u>

Since these two equations $\begin{aligned} 4x + 2y &= 2x \\ x + 3y &= 2y \end{aligned}$ are dependent, we can use either one to determine the straight line of vectors (called eigenvectors) in the phase plane. In this case, straight line solutions are found along the line $y = -x$.

Any solution along this line can therefore be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $\lambda = 5$

Since these two equations $\begin{aligned} 4x + 2y &= 5x \\ x + 3y &= 5y \end{aligned}$ are dependent, we can use either one to determine the straight line of vectors (called eigenvectors) in the phase plane. In this case, straight line solutions are found along the line $y = \frac{1}{2}x$.

Any solution along this line can therefore be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution is therefore

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

6. In the previous example the general solution was determined to be

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

What is the specific solution for the initial condition $(-3, -2)$? Without using technology, sketch the graph of this solution in the phase plane (for $t \to \infty$ and as $t \to -\infty$) and explain how you figured out what the graph looks like based on the equations for the solution.

$$-3 = k_1 (1)(1) + k_2 (1)(2)$$
$$-2 = k_1 (1)(-1) + k_2 (1)(1)$$

$$k_1 + 2k_2 = -3$$
$$-k_1 + k_2 = -2$$

$$3k_2 = -5$$
$$k_2 = -\frac{5}{3}$$

$$k_1 + 2\left(-\frac{5}{3}\right) = -3$$

$$k_1 = -3 + \frac{10}{3} = \frac{1}{3}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{3} e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + -\frac{5}{3} e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$