Second Order Linear Differential Equations

A second order linear differential equation has the form

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = G(t)$$

where P, Q, R, and G are continuous functions. There are many applications for which this type of differential equation is a useful model. Your previous work with the spring mass problem was one such example. Here are some other examples.

Glass Breaking: You probably have all seen in cartoons or on Mythbusters where a wineglass is broken by singing a particular high-pitched note. The phenomenon that makes this possible is called *resonance.* Resonance results from the fact that the crystalline structures of certain solids have natural frequencies of vibration. An external force of the same frequency will "resonate" with the object and create a huge increase in energy. For instance, if the frequency of a musical note matches the natural vibration of a crystal wineglass, the glass will vibrate with increasing amplitude until it shatters. The following is one model for understanding resonance:

$$\frac{d^2x}{dt^2} + k^2x = \cos(kt)$$

Tacoma Narrows Bridge: The Tacoma Narrows Bridge in Washington State was one of the largest suspended bridges built at the time. The bridge connecting the Tacoma Narrows channel collapsed in a dramatic way on Thursday November 7, 1940. Winds of 35-46 miles/hours produced an oscillation which eventually broke the construction. The bridge began first to vibrate torsionally, giving it a twisting motion. Later the vibrations entered a natural resonance (same term as in the glass breaking) with the bridge. Here is a simplified second order differential equation that models the situation of the Tacoma Bridge:

$$\frac{d^2y}{dt^2} + 4y = 2\sin(2.1t)$$

Sometimes resonance is a good thing! Violins, for instance, are designed so that their body resonates at as many different frequencies as possible, which allows you to hear the vibrations of the strings!

There are many other situations that can be modeled with second order differential equations, including RLC circuits, pendulums, car springs bouncing, etc. In this section you will learn how to solve second order linear differential equations with constant coefficients. That is, equations where P, Q, and R are constant. If G is zero, then the equation is called **homogeneous**. When G is nonzero then the equation is called **nonhomogeneous**. As you will discover in the problems that follow, the distinction between homogeneous and non-homogeneous equations will be quite useful.

Guess and Test

1. (a) Read the following equations *with meaning*, by completing the following sentence, "$x(t)$ is a function for which its second derivative ..." (try saying "itself" instead of "x").

... the negative of itself. ... plus 4 times itself is 0

i. $\frac{d^2x}{dt^2} = -x$ ii. $\frac{d^2x}{dt^2} + x = 0$ iii. $\frac{d^2x}{dt^2} + 4x = 0$ iv. $\frac{d^2x}{dt^2} = x$ ... itself

(b) For each differential equation above, based on your readings *with meaning*, find two different solution functions.

$X = \cos t$
$X = \sin t$
i & ii

iii. $X = \sin 2t$
$X = \cos 2t$

iv. $e^t = X$

2. Your task in this problem is to use the "guess and test" approach to find a solution to the linear second order, homogeneous differential equation

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 0$$

By now you know very well that solutions are functions. What is your best guess for a function whose second derivative plus 10 times its first derivative plus 9 times the function itself sum to zero? Explain briefly the rationale for your guess and then test it out to see if it works. If it doesn't work keep trying.

$X = e^{-9t}$?

$X = e^{-t}$ deriv flips sign, and then again

meaning $e^{-t} + 9e^{-t} + 10(-e^{-t}) = 0$ ✓

$\frac{d^2x}{dt} + 9X + 10\left(\frac{dx}{dt}\right) = 0$

3. Determine if a constant multiple of your solution is also a solution.

$X = Ae^{-t}$
$X' = -Ae^{-t}$
$X'' = Ae^{-t}$

$Ae^{-t} + 10(-Ae^{-t}) + 9Ae^{-t} = 0$ ✓

4. Try and find a different solution, one that is not a constant multiple of your solution to problem 2.

$x = e^{-9t}$

$x' = -9e^{-9t}$

$x'' = 81e^{-9t}$

$81e^{-9t} + (-90)e^{-9t} + 9e^{-9t} = 0$ ✓

5. Determine the *general solution* to $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 0$.

$x = c_1 e^{-t} + c_2 e^{-9t}$

6. Consider again the differential equation $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 0$.

By guessing $x(t) = e^{rt}$, show that this guess yields a solution to the differential equation precisely when $r^2+10r+9=0$. $x = e^{rt}$ $x' = re^{rt}$ $x'' = r^2e^{rt}$	$r^2e^{rt} + 10re^{rt} + 9e^{rt} = 0$ $e^{rt}(r^2 + 10r + 9) = 0$ $(r+9)(r+1) = 0$
Solve this quadratic equation to find two different values of r.	$r = -9, \ r = -1$
State two different solutions for the differential equation, one for each value of r.	$x_1 = e^{-9t}$ $\quad x_2 = e^{-t}$
Form the general solution by multiplying your two solutions by constants c_1 and c_2, and adding the results.	$x(t) = c_1e^{-9t} + c_2e^{-t}$
Congratulate yourself :)	yea!

7. Find the general solution to the following differential equation: $\frac{d^2x}{dt^2} + \frac{dx}{dt} - 6x = 0$.

$r^2e^{rt} + re^{rt} - 6e^{rt} = 0$

$e^{rt}(r^2 + r - 6) = 0$

$(r+3)(r-2) = 0$

$r = -3, \ r = 2$

$x = c_1e^{-3t} + c_2e^{2t}$

The Nonhomogeneous Case

8. In this next problem your task is to find a solution to the following **nonhomogeneous** version of the differential equation from the first problem:

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 18.$$

What is your best guess for a function whose second derivative plus 10 times its first derivative plus 9 times the function itself sum to 18? Test out your guess to see if it works. If it doesn't work keep trying.

a constant

$x = A$

$x' = 0$ $\qquad 0 + 10(0) + 9(A) = 18$

$x'' = 0$

$9A = 18$

$A = 2$

$x = 2$

The solution you found in the previous problem is called the **particular solution** to the nonhomogeneous differential equation. To find the general solution to the nonhomogeneous differential equation you simply add the particular solution to the general solution to the corresponding homogeneous equation. This 3-step strategy (1 - Find the general solution to the corresponding homogeneous equation; 2 - Find the particular solution to the nonhomogeneous equation, 3 - Add the previous results) is called the **Method of Undetermined Coefficients**.

9. Write down the general solution to $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 18$ and give a convincing argument for why this sum is in fact a solution is the nonhomogeneous differential equation.

principle of superposition – since homogeneous solution is zero, adding in new nonhomogeneous solution covers all possible cases.

$X(t) = c_1 e^{-9t} + c_2 e^{-t} + 2$

10. Sean and Phil are trying to find the particular solution to $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 85\sin(2t)$. Sean guesses $x(t) = A\sin(2t)$ for the particular solution and Phil guesses $x(t) = B\cos(2t)$.

   (a) Do you think these are reasonable guesses? Explain why or why not.

   individually, they won't work since if we just use $A\sin 2t$ the derivative will produce a cosine, and vice versa. But the two together might work. (should)

   (b) For each of their guesses, can you find a value of A or B such that their guess is a solution? If yes, write down the general solution. If no, come up with a different guess for the particular solution and show that your guess is correct.

$X = A\sin 2t + B\cos 2t$

$X' = 2A\cos 2t - 2B\sin 2t$

$X'' = -4A\sin 2t - 4B\cos 2t$

$5A - 20B = 85$

$20A + 5B = 0$

$A = 1,\ B = -4$

$-4A\sin 2t - 4B\cos 2t + 10(2A\cos 2t - 2B\sin 2t) + 9(A\sin 2t + B\cos 2t) = 85\sin 2t$

$\sin 2t(-4A - 20B + 9A) = 85\sin 2t$

$\cos 2t(-4B + 20A + 9B) = 0\cos 2t$

11. Write down the general solution to $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + 9x = 85\sin(2t)$.

$$X(t) = c_1e^{-9t} + c_2e^{-t} + \sin 2t - 4\cos 2t$$

12. Find the general solution to $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + 9x = 85\sin(2t)+18$. Explain why you can do this by combining results from the previous problems.

$$X(t) = c_1e^{-9t} + c_2e^{-t} + \sin 2t - 4\cos 2t + 2$$

2 will only solve for the 18, the sine & cosine only solves for the sine. to get both, we need both solutions

13. An aside on complex numbers:

(a) Show that $x(s) = e^{is}$ and $x(s) = \cos(s) + i\sin(s)$ are both solutions to the differential equation $dx/ds = ix$ with $x(0) = 1$. What does the uniqueness theorem imply about these two solutions?

$X(s) = e^{is}$

$X'(s) = ie^{is}$

$ie^{is} = i(e^{is})$

$e^{i(0)} = 1$ ✓

$X(s) = \cos(s) + i\sin(s)$

$X'(s) = -\sin(s) + i\cos(s)$

$i(X(s)) = i\cos(s) + i^2\sin(s) =$

$i\cos(s) - 1\sin(s)$

$X(0) = (1) + i(0) = 1$ ✓

they are the same

(b) The above result is called Euler's formula. Multiplying by $e^{\alpha t}$ and using $s = \beta t$, we can rewrite the formula into the following form: $e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$. Use this to find a similar formula for $e^{(\alpha-\beta i)t}$.

$$e^{(\alpha-\beta i)t} = e^{at}(\cos(-\beta t) + i\sin(-\beta t)) =$$

$$e^{at}(\cos(\beta t) - i\sin(\beta t))$$

(c) Suppose you have two functions:

$$A(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$
$$B(t) = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

Simplify the following expressions in (i) and (ii) then answer (iii) and (iv).

i. $x_1(t) = \dfrac{A(t) + B(t)}{2}$

$$\frac{e^{at}(2\cos\beta t)}{2} = e^{\alpha t}\cos\beta t$$

ii. $x_2(t) = i\dfrac{A(t) - B(t)}{2}$

$$i\,\frac{e^{\alpha t}(2i\sin\beta t)}{2} = -e^{\alpha t}\sin\beta t$$

iii. What do you notice about your solutions in (i) and (ii), compared to $A(t)$ and $B(t)$?

they are real

iv. If $A(t)$ and $B(t)$ were solutions to a differential equation of the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0,$$

would $x_1(t)$ and $x_2(t)$ be solutions too? How about $c_1x_1(t) + c_2x_2(t)$ for arbitrary constants c_1 and c_2?

yes.

14. Find the general solution to the homogeneous differential equation

$$\frac{d^2x}{dt^2} + 25x = 0$$

You will find that your guess results in complex roots to the quadratic. Use the above results on exponentiation of complex numbers to find the general solution to the differential equation.

$r^2e^{rt} + 25e^{rt} = 0$

$e^{2t}(r^2+25) = 0$

$r = \pm 5i$

$c_1 e^{-5it} + c_2 e^{5it} = x(t)$

$x(t) = c_1 \cos 5t + c_2 \sin 5t$

15. (a) Consider the nonhomogeneous differential equation

$$\frac{d^2x}{dt^2} + 25x = 10\cos(5t)$$

Suppose you wish to find the particular solution to this differential equation. Explain why a guess of the form $x(t) = A\cos(5t) + B\sin(5t)$ is doomed to fail.

when we take derivative of this guess it gives 0 not $10 \cos 5t$

(b) Nevertheless, explain why your particular solution must have terms that *look like* $\cos(5t)$ and $\sin(5t)$.

because we need terms like $\cos 5t$ to survive after putting in equation

(c) For an unknown differentiable function $f(t)$, write down the first and second derivatives of $tf(t)$, what do you notice?

$\frac{d}{dt}[t f(t)] = f(t) + t f'(t)$

(d) Explain why a guess of $At\cos(5t)$ is insufficient to find the particular solution.

for the same reason we need both sine and cosine

(e) Use the guess $x(t) = t(A\cos(5t) + B\sin(5t))$ to find a particular solution to the above equation.

$X'(t) = (A\cos 5t + B\sin 5t) + t(-5A\sin 5t + 5B\cos 5t)$

$X''(t) = -5A\sin 5t + B5\cos 5t + (-5A\sin 5t + 5B\cos 5t) +$

$t(-25A\cos 5t - 25B\sin 5t)$

$-10A\sin 5t + 10B\cos 5t + \cancel{t(-25A\cos 5t - 25B\sin 5t) + 25t(A\cos 5t + B\sin 5t)} = 10\cos 5t$

$A = 0,\ B = 1$

16. (a) Find the general solution to

$$\frac{d^2x}{dt^2} + 25x = 10\cos(5t).$$

$X(t) = C_1\sin 5t + C_2\cos 5t + t\cos 5t$

(b) Find the specific solution for initial conditions $x(0) = 0$, $x'(0) = 1$.

$X(0) = C_1(0) + C_2(1) + 0(1) = 0 \qquad C_2 = 0$

$X'(0) = 5C_1(1) + (1) + (0)(0) = 1$

$\cos 5t \qquad \cos 5t - 5t\sin t$

$5C_1 = 1 \rightarrow C_1 = \frac{1}{5}$

$X(t) = \frac{1}{5}\sin 5t + t\cos 5t$