Bees and Flowers

Often scientists use rate of change equations in their study of population growth for one or more species. In this problem we study systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is, both species are *harmed by* interaction) or cooperative (that is, both species *benefit from* interaction).

1. Which system of rate of change equations below describes a situation where the two species compete and which system describes cooperative species? Explain your reasoning.

(i) $\frac{dx}{dt} = -5x + 2xy$, $\frac{dy}{dt} = -4y + 3xy$

(ii) $\frac{dx}{dt} = 4x - 2xy$, $\frac{dy}{dt} = 2y - xy$

if $y=0$ $\frac{dx}{dt} = -5x$
if $x=0$ $\frac{dy}{dt} = -4y$
both decrease over time

if $y=5$ $\frac{dx}{dt} = -5x + 2x(5) = -5x + 10x = 5x$
if $x=5$ $\frac{dy}{dt} = -4y + 3y(5) = -4y + 15y = 11y$
both increase over time

Cooperation model
for large enough populations both populations grow
alone, there is die-off.

if $y=0$ $\frac{dx}{dt} = 4x$
if $x=0$ $\frac{dy}{dt} = 2y$
both grow

if $y=5$ $\frac{dy}{dt} = 4x - 2x(5) = 4x - 10x = -6x$
if $x=5$ $\frac{dy}{dt} = 2y - 5y = -3y$
both decay/decrease over time

Competition model
for large enough populations they compete and contribute to the other population's decline

Sign of interaction (xy) term determines behaviour

A Simplified Situation

The previous problem dealt with a complex situation with two interacting species. To develop the ideas and tools that we will need to further analyze complex situations like these, we will simplify the situation by making the following assumptions:

- There is only one species (*e.g.*, fish)
- The species has been in its habitat (*e.g.*, a lake) for some time prior to what we call $t = 0$
- The species has access to unlimited resources (*e.g.*, food, space, water)
- The species reproduces continuously

} exponential growth

2. Given these assumptions for a certain lake containing fish, sketch three possible population versus time graphs: one starting at $P = 10$, one starting at $P = 20$, and the third starting at $P = 30$.

(a) For your graph starting with $P = 10$, how does the slope vary as time increases? Explain.

it increases, exponentially

(b) For a set P value, say $P = 30$, how do the slopes vary across the three graphs you drew?

the slopes are the same

3. This situation can also be modeled with a rate of change equation, $\frac{dP}{dt} = \textit{something}$. What should the "something" be? Should the rate of change be stated in terms of just P, just t, or both P and t? Make a conjecture about the right hand side of the rate of change equation and provide reasons for your conjecture.

just P (because of 2b above)

$$\frac{dP}{dt} = kP$$

What Exactly is a Differential Equation and What are Solutions?

A **differential equation** is an equation that relates an unknown function to its derivative(s). Suppose $y = y(t)$ is some unknown function, then a differential equation, or rate of change equation, would express the rate of change, $\frac{dy}{dt}$, in terms of y and/or t. For example, all of the following are differential equations.

$$\frac{dP}{dt} = kP, \qquad \frac{dy}{dt} = y + 2t, \qquad \frac{dy}{dt} = t^2 + 5, \qquad \frac{dy}{dt} = \frac{6y-2}{ty}, \qquad \frac{dy}{dt} = \frac{y^2-1}{t^2+2t}$$

In particular, these are all examples of *first order* differential equations because only the first derivative appears in the equation. Given a rate of change equation for some unknown function, **solutions** to this rate of change equation are *functions* that satisfies the rate change equation. A constant function that satisfies the differential equation is called an **equilibrium solution**.

One way to read the differential equation $\frac{dy}{dt} = y + 2t$ aloud you would say, "dee y dee t equals y plus two times t." However, this does **not** relate to the *meaning* of the solution. How might you read this differential equation *with meaning*?

4. (a) Is the function $y = 1 + t$ a solution to the differential equation $\frac{dy}{dt} = \frac{y^2-1}{t^2+2t}$? How about the function $y = 1 + 2t$? How about $y = 1$? Explain your reasoning.

$y = 1+t$ yes
$\frac{dy}{dt} = 1$
$1 \overset{?}{=} \frac{(1+t)^2-1}{t^2+2t} \overset{?}{=} \frac{t^2+2t+1-1}{t^2+2t} = 1$ ✓

$y = 1+2t$ no
$\frac{dy}{dt} = 2$
$2 \overset{?}{=} \frac{(1+2t)^2-1}{t^2+2t} \overset{?}{=} \frac{4t^2+4t+1-1}{t^2+2t} = \frac{4(t^2+t)}{t^2+2t} \neq 2$

$y = 1$ yes
$\frac{dy}{dt} = 0$
$0 \overset{?}{=} \frac{1^2-1}{t^2+2t} = \frac{0}{t^2+2t} = 0$ ✓

(b) Is the function $y = t^3 + 2t$ a solution to the differential equation $\frac{dy}{dt} = 3y^2 + 2$? Why or why not?

$y = t^3+2t$
$\frac{dy}{dt} = 3t^2+2$

$3t^2+2 \overset{?}{=} 3(t^2+2t)^2+2$
$\overset{?}{=} 3(t^4+4t^3+4t^2)+2$
$3t^2+2 \neq 3t^4+12t^3+12t^2+2$

they are not equal; it is not a solution; it does not satisfy the equation

5. Figure out all the functions that satisfy the rate of change equation $\frac{dP}{dt} = 0.3P$. (*Hint*: read the differential equation with meaning.)

→ what function's derivative is 0.3 times the original function?

$P = Ae^{0.3t}$
$\frac{dP}{dt} = 0.3(Ae^{0.3t}) = 0.3P$

exponential $e^{0.3t}$ and any multiple

6. Figure out all of the solutions to the differential equation $\frac{dy}{dt} = t^2 + 5$.

$\int dy = \int t^2+5 \, dt \Rightarrow$
$y = \frac{1}{3}t^3+5t+C$

function of t-only
integrate

Slope Fields

A **slope field** is a graphical representation of a rate of change equation. Given a rate of change equation, if we plug in particular values of (t, y) then $\frac{dy}{dt}$ tells you the slope of the tangent vector to the solution at that point.

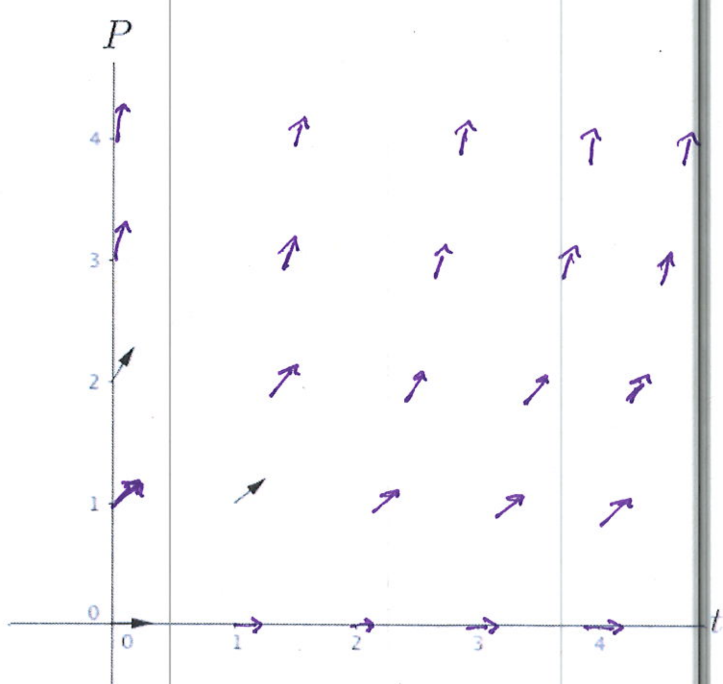
For example, consider the rate of change equation $\frac{dy}{dt} = y + 2t$. At the point (1, 3), the value of $\frac{dy}{dt}$ is 5. Thus, the slope field for this equation would show a vector at the point (1, 3) with slope 5. A slope field depicts the exact slope of many such vectors, where we take each vector to be uniform length. Slope fields are useful because they provide a graphical approach for obtaining qualitatively correct graphs of the functions that satisfy a differential equation.

7. Below is a partially completed slope field for $\frac{dP}{dt} = 0.8P$.

   (a) Plot many more tangent vectors to create a slope field.

   (b) Use your slope field to sketch in qualitatively correct graphs of the solution functions that start at $P = 0, 1,$ and 4, respectively. Note: the value of P at an initial time (typically $t = 0$) is called an **initial condition**.

   (c) Recall that a solution to a differential equation is a function that satisfies the differential equation. Explain how the graph with initial condition $P(0) = 1$ can graphically be thought of as a solution to the differential equation when the differential equation is represented by its slope field.

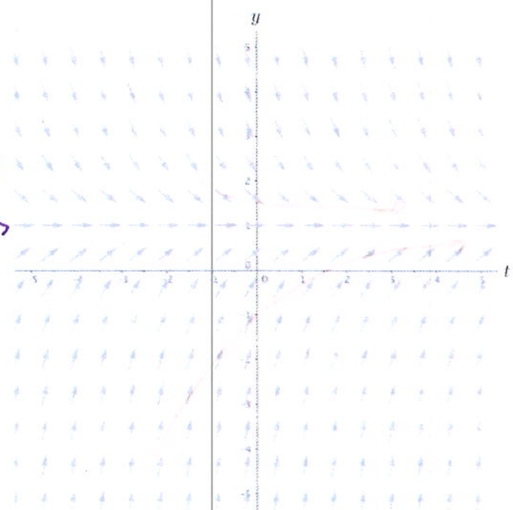


t	P	$\frac{dP}{dt} = 0.8P$
0	0	0
0	2	1.6
1	1	0.8
1	0	0
0	1	.8
1	2	1.6
0	3	2.4
0	4	3.2
2	2	1.6
2	4	3.2

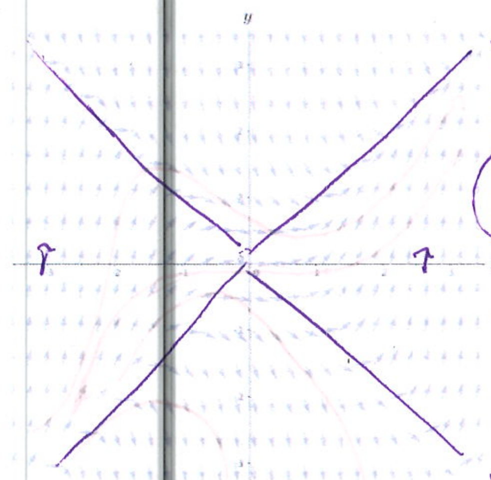
8. Below are seven rate of changes equations and three different slope fields. Without using technology, identify which differential equation is the best match for each slope field (thus you will have four rate of change equations left over). Explain your reasoning.

$$\text{(i) } \frac{dy}{dt} = t-1 \quad \text{(ii) } \frac{dy}{dt} = 1-y^2 \quad \text{(iii) } \frac{dy}{dt} = y^2-t^2 \quad \text{(iv) } \frac{dy}{dt} = 1-y$$

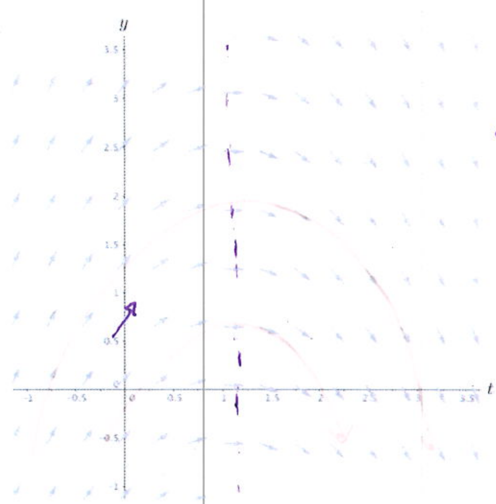
$$\text{(v) } \frac{dy}{dt} = t^2-y^2 \quad \text{(vi) } \frac{dy}{dt} = 1-t \quad \text{(vii) } \frac{dy}{dt} = 9t^2-y^2$$



(a)



(b)



(c)

9. For each of the slope fields in the previous problem, sketch in graphs of several different qualitatively correct solutions.