Proposed Paths of Descent

A group of scientists at the Federal Aviation Association has come up with the following two different rate of change equations to predict the height of a helicopter as it nears the ground:

$$\frac{dh}{dt} = -h \qquad \text{and} \qquad \frac{dh}{dt} = -h^{\frac{1}{3}}$$

For both rate of change equations h is in feet and t is in minutes. The scientists, of course, want their models to predict that a helicopter actually lands - but do either or both of the proposed models predict this?

1. Getting familiar with the differential equations:

   (a) Just by examining the rate of change equations, what can you say about the height of the helicopter as predicted by $\frac{dh}{dt} = -h$ and by $\frac{dh}{dt} = -h^{\frac{1}{3}}$? More specifically, as h approaches zero, what can you say about $\frac{dh}{dt}$ and what does that imply about whether the model predicts that the helicopter lands?

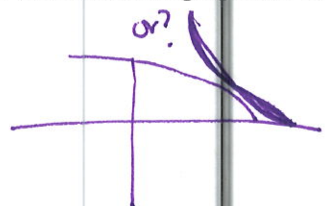
   $\frac{dh}{dt} = -h$ is exponential decay. it never lands

   as $h \to 0$ $\left|\frac{dh}{dt}\right| > h$. this will land

   (b) Sketch your best guess for a height versus time solution graph for each rate of change equation.

   or?

2. (a) What do each of the proposed rate of change equations say about the solution to the differential equation if the helicopter is already on the ground? Explain and sketch a corresponding graph of height versus time on the same set of axes from part 1b.

   stays on ground

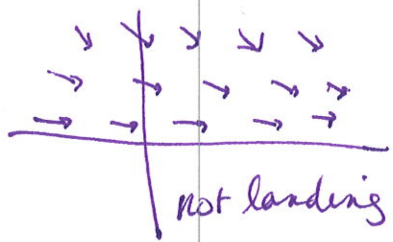
   

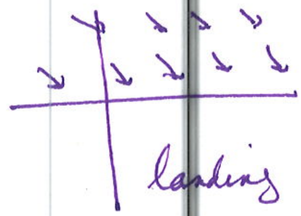
   or

   (b) Interpret the initial condition $h(0) = 0$, and explain why $h(t) = 0$ should be a solution to each differential equation under this initial condition.

   the height is zero at $t=0$
   the rate of change is zero and so
   it could stay on ground forever

3. Use the Geogebra applet, https://ggbm.at/dJsACfAN, to investigate the slope fields. What do the slope fields suggest about whether the model predicts if the helicopters will land? How do the slope fields compare with your sketches from part 1b?





even as you zoom in, the first equation slopes are nearly horizontal, but second equation still distinctly negative / pointing down → toward a landing

4. Solve the following initial value problems:

(a) $\frac{dh}{dt} = -h$

(i) $h(0) = 2$ (ii) $h(0) = 0$ (*Hint*: Use problem 2b)

$\int \frac{dh}{h} = \int -dt$

$\ln h = -t + C$

$h = e^{-t} \cdot k$

$h = 2e^{-t}$

$h = 0$

(b) $\frac{dh}{dt} = -h^{\frac{1}{3}}$

(i) $h(0) = 2$ (ii) $h(0) = 0$ (*Hint*: Use problem 2b)

$\int \frac{dh}{h^{1/3}} = \int -dt$

$\int h^{-1/3} dh =$

$\frac{3}{2} h^{2/3} = -t + C$

$h^{2/3} = -\frac{2}{3}t + C$

$h = \pm\left(-\frac{2}{3}t + C\right)^{3/2}$

$2 = \left(-\frac{2}{3}\cancel{0} + C\right)^{3/2}$

$\sqrt[3]{4} = C$

$h = \left(-\frac{2}{3}t + \sqrt[3]{4}\right)^{3/2}$

$0 = \left(-\frac{2}{3}(0) + C\right)^{3/2}$

$C = 0$

$h = \left(-\frac{2}{3}t\right)^{3/2}$ (t must be negative)

5. (a) For each differential equation, interpret the results from problem 4 in terms of whether the model predicts the helicopter will ever touch the ground. If so, at what time?

$h = 2e^{-t}$ never lands

$h = 0$ stays on ground

$h = \left(-\frac{2}{3}t + \sqrt[3]{4}\right)^{3/2}$

touches ground

$+\frac{2}{3}t = \sqrt[3]{4}$ $t \approx 2.38$

$h = \left(-\frac{2}{3}t\right)^{3/2}$

on ground now

takes off in past time

(b) For each differential equation, interpret the results from problem 4 in terms of whether graphs of (i) and (ii) will ever touch or cross.

$h = 2e^{-t}$

$h = 0$

never touches or crosses

touches, each one consists of 2 solutions in h

a positive and a negative solution

6. One difference between the two differential equations is the partial derivative of the right hand side at $h = 0$. That is,

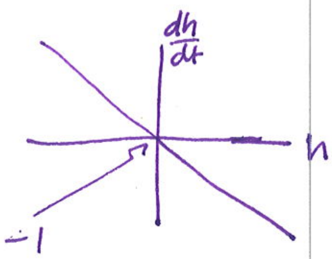
$$\frac{\partial f}{\partial h}, \quad \text{where } f(h) = -h$$

for one differential equation is different than

$$\frac{\partial f}{\partial h}, \quad \text{where } f(h) = -h^{\frac{1}{3}}$$

for the other differential equation.

Accurately draw graphs of $\frac{dh}{dt}$ versus h for both differential equations and use these graphs to determine the partial derivatives at $h = 0$ for each differential equation.



undefined
slope is vertical

The Uniqueness Theorem

In the formal language of differential equations, the term "unique" or "uniqueness" refers to whether or not two solution functions ever touch or cross each other. Using this terminology, the two solutions you found to $\frac{dh}{dt} = -h$ are unique while the two solutions you found to $\frac{dh}{dt} = h^{\frac{1}{3}}$ are not unique. Fortunately, one does not have to always analytically solve a differential equation to determine if solutions will or will not be unique. There is a theorem, the **Uniqueness Theorem**, which sets out conditions for when solutions are unique.

Theorem. Let $f(x, y)$ be a real valued function which is continuous on the rectangle

$$R = \{(x, y) : |x - x_0| \leq a, |y - y_0| \leq b\}.$$

Assume f has a partial derivative with respect to y and that this partial derivative $\partial f / \partial y$ is also continuous on the rectangle R. Then there exists an interval

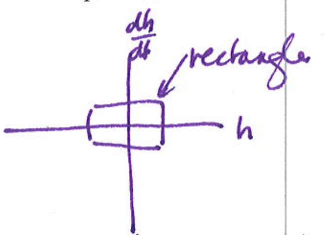
$$I = [x_0 - h, x_0 + h] \text{ (with } h \leq a)$$

such that the initial value problem

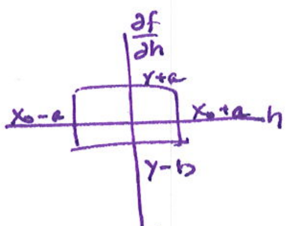
$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

has a unique solution $y(x)$ defined on the interval I.

7. Explain how the conditions of this theorem relate to solutions of $\frac{dh}{dt} = -h$.



Continuous



Continuous

$-h$ i.e. $y = -x$ is continuous inside a rectangle around $x=0$. and $\frac{\partial f}{\partial h} = -1$ which is also continuous around any rectangle near $x=0$ → Thus, unique Solution

8. If you are given a differential equation and determine that the conditions of the uniqueness theorem are NOT met in a specific range of y-values, what can you conclude about the graphs of solution functions within that range of y-values? Explain.

either no solution exists at that point, or the solutions are not unique