MTH 266, Exam #2, Part I, Spring 2019 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available. On this portion of the exam, you may **NOT** use a calculator.

1. Compute the determinant by the cofactor method. (15 points)

$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 1 \\ \cancel{3} & \cancel{1} & \cancel{5} & \cancel{-1} \end{vmatrix}$$

$$-5\begin{vmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{vmatrix} = 5\left[2\begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}\right] =$$

$$5\left[2(2-2)+1(1-1)\right] = 5(0+0) = 0$$

2. Compute the determinant by using row operations. (10 points)

$$\begin{vmatrix} 0 & 3 & -1 & 5 \\ 1 & 0 & -2 & 4 \\ -3 & 2 & 1 & -3 \\ 0 & 5 & 2 & 3 \end{vmatrix}$$

$3R_2 + R_3 \to R_3$

$$\begin{vmatrix} 0 & 3 & -1 & 5 \\ \cancel{1} & \cancel{0} & \cancel{-2} & \cancel{4} \\ 0 & 2 & -5 & 9 \\ 0 & 5 & 2 & 3 \end{vmatrix} =$$

$$-1\begin{vmatrix} 3 & -1 & 5 \\ 2 & -5 & 9 \\ 5 & 2 & 3 \end{vmatrix}$$

$-5R_1 + R_2 \to R_2$
$2R_1 + R_3 \to R_3$

$$-1\begin{vmatrix} \cancel{3} & \cancel{-1} & \cancel{5} \\ -13 & 0 & -16 \\ 11 & 0 & 13 \end{vmatrix} =$$

$$(-1)[-(-1)]\begin{vmatrix} -13 & -16 \\ 11 & 13 \end{vmatrix} = -1(-169+176) = -1(7) = -7$$

3. Determine if the following sets are linearly independent or dependent.
Justify your answers **without performing matrix calculations.** (4 points each)

a. $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ -4 \\ 12 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -6 \\ -19 \end{bmatrix} \right\}$

dependent
5 vectors in $\mathbb{R}^4$
always dependent

b. $\left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} \right\}$

independent

4. Given that A and B are $n \times n$ matrices with det A = -7 and det B = -2, find the following. (4 points each)

a) det (AB)

14

b) det (A^{-1})

$-\frac{1}{7}$

c) det ($-AB^6$)

$(-1)^n(-7)(-2)^6$

$(-1)^{n+1}448$

d) det (B^T)

-2

e) det (5A)

$5^n(-7)$

f) det ($A^{-1}BA$)

-2

5. Determine if each statement is True or False. (3 points each)

a. T F If matrix B is formed by multiplying a row of matrix A by -1, then det B = -det A

only if dimensions are odd

b. T F The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution when there is at least one free variable. no free variables

c. T F If an $m \times n$ matrix has a pivot in every row, then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each **b** in R^m. solution exists

d. T F If $\{\vec{u}, \vec{v}, \vec{w}, \vec{x}\}$ is linearly independent, then $\vec{u}, \vec{v}, \vec{w}, and\ \vec{x}$ are not in R^3.

e. T F If A and B are $m \times n$ matrices, then both AB^{T} and $A^{T}B$ are defined.

f. T F Interchanging three rows of an $n \times n$ matrix A, you will not change the determinant. 2 row interchanges

g. T F If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, then so is $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p+1}\}$.

p-1 yes. not more

h. T F The pivot columns of a matrix are always linearly dependent.

i. T F If det A is zero, then two rows or two columns of A are the same, or a row or a column is zero. Can be multiples or linear combinations

j. T F If A and B are row equivalent, then their column spaces are the same.

Same dimension only

k. T F The vector space P_4 and R^5 are isomorphic.

l. T F A linearly independent set in a subspace H is a basis for H. must also span

m. T F If P_B is the change-of-coordinates matrix, then $[\vec{x}]_B = P_B^{-1}\vec{x}$ for $\vec{x}$ in V.

n. T F There are only two conditions a vector space must satisfy: it must be closed under addition and closed under multiplication. and include $\vec{0}$

Scalar

o. T F The vector space of 2x3 matrices is isomorphic to $\mathbb{R}^6$.

p. T F The nullspace of an mxn matrix A is a subspace of $\mathbb{R}^m$.

n

q. T F $(AB)^{-1} = A^{-1}B^{-1}$ $B^{-1}A^{-1}$

r. T F The change of basis matrix is constructed from putting the basis vectors into the rows of P_B.

6. Suppose matrix A is a 7x9 matrix with 5 pivot columns. Determine the following. (12 points)

dim Col A = 5 dim Nul A = 4

dim Row A = 5 If Col A is a subspace of R^m, then m = 7

Rank A = 5 If Nul A is a subspace of R^n, then n = 9

MTH 266, Exam #2, Part II, Spring 2019 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available. On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

1. Determine if the columns of $A=\begin{bmatrix} 1 & 5 & 2 \\ 6 & 4 & -1 \\ -4 & -2 & 1 \\ 3 & 1 & -1 \end{bmatrix}$ form a linearly independent set and justify your answer. (6 points)

no, they are not rref $\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

no pivot in column three

2. Given $T:\mathbb{R}^4 \to \mathbb{R}^3$ such that $T(\mathbf{x})=\begin{bmatrix} x_1-4x_3+x_4 \\ x_2+2x_3 \\ -x_1+5x_4 \end{bmatrix}$ answer the following.

   a. Find the standard matrix, *A*, such that $T(\mathbf{x})=A\mathbf{x}$. (5 points)

$$\begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 2 & 0 \\ -1 & 0 & 0 & 5 \end{bmatrix}$$

   b. Is *T* onto $\mathbb{R}^3$? Justify your answer. (4 points)

yes, since there is a pivot in every row

$$\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3/2 \end{bmatrix}$$

c. Is T one-to-one? Justify your answer. (4 points)

no, since there is not a pivot in every column (projects $\mathbb{R}^4$ onto $\mathbb{R}^3$)

3. Determine if the set $H = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^3$. Justify your answer. (6 points)

yes. reduces to the identity

4. Assume that $A = \begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 2 & 4 & -5 & 1 & 2 \\ 1 & 2 & 0 & 3 & 1 \\ 3 & 6 & -1 & 8 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & -1 & 8 & 1 \\ 0 & 0 & 13 & 13 & -4 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ are row equivalent.

a. Find a basis for the column space of *A*. (6 points)

$$\text{Col } A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

b. Find a basis for the null space of *A*. (8 points)

$$\begin{aligned} x_1 &= -\tfrac{2}{3}x_2 - 3x_4 \\ x_2 &= x_2 \\ x_3 &= -x_4 \\ x_4 &= x_4 \\ x_5 &= 0 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_3$$

$$\text{Nul } A = \left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

c. Determine if $\mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 4 \\ -3 \end{bmatrix}$ is in Col A. Show appropriate work to justify your answer. (6 points)

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 2 & -5 & 2 & 0 \\ 1 & 0 & 1 & 4 \\ 3 & 1 & 1 & -3 \end{array}\right] \text{rref} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\vec{b}$ is not in Col A. System is inconsistent

5. Given the basis B = $\{1-2t^3, t-2t^2, 2-5t+t^2, 3-t^2+7t^3\}$ for P_3. Find $\vec{p}(t) = 6+19t-7t^2$ in this basis. (15 points)

$$P = \begin{bmatrix} 6 \\ 19 \\ -7 \\ 0 \end{bmatrix}$$

$$P_B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & 0 \\ 0 & -2 & 1 & -1 \\ -2 & 0 & 0 & 7 \end{bmatrix}$$

$$P_B^{-1}\vec{p} = \begin{bmatrix} 63/113 & 28/113 & 14/113 & -25/113 \\ -10/113 & -17/113 & -65/113 & -5/113 \\ -2/113 & -26/113 & -13/113 & -1/113 \\ 18/113 & 8/113 & 4/113 & -9/113 \end{bmatrix} \begin{bmatrix} 6 \\ 19 \\ -7 \\ 0 \end{bmatrix} = \begin{bmatrix} 812/113 \\ 72/113 \\ -415/113 \\ 232/113 \end{bmatrix}_B$$

$$p(t)_B = 812/113(1-2t^3) + 72/113(t-2t^2) - 415/113(2-5t+t^2) + 232/113(3-t^2+7t^3)$$

6. Given that $\det A^{-1} = \dfrac{1}{\det A}$ if A is invertible, use this fact and the fact that AB is invertible to prove that both A and B must be invertible. [Hint: use multiplication properties of the determinant and what you know about nxn identity matrices.] (10 points)

$$AA^{-1} = I$$
$$\det(AA^{-1}) = \det I$$
$$\det A \cdot \det A^{-1} = 1$$
$$\det A^{-1} = \frac{1}{\det A}$$

7. Prove that the following are vector spaces or show that they are not. (6 points each)

a. $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a = b + c; a, b, c \text{ real} \right\}$

$\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \vec{y} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$ $d = e + f$

$\vec{x} + \vec{y} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$ $a + d \stackrel{?}{=} (b+e) + (c+f)$

$a + d = (b+c) + (e+f)$ ✓

$k\vec{x} = \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}$

$ka \stackrel{?}{=} kb + kc =$

$k(b+c) = ka$ ✓

$\vec{0}$ in set if $a = b = c = 0$

is a subspace

b. $W = \left\{ \begin{bmatrix} a & a+2 \\ b & c \end{bmatrix}, a, b, c \text{ real} \right\}$

is not a subspace

if $a = b = c = 0$ Then $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ if $b = c = 0$, $a = -2$, then $\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$

no $\vec{0}$ so not a subspace

8. Given the bases B={b_1, b_2, b_3} and C={c_1, c_2, c_3} below, find the change of basis matrices $\underset{C \leftarrow B}{P}$ and $\underset{B \leftarrow C}{P}$. If the B-coordinate vector for $\vec{x}$ is as shown, find the C-coordinate vector for $\vec{x}$. (15 points)

$$\vec{b_1} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 1 \\ 6 \\ -5 \end{bmatrix}, \vec{b_3} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \vec{c_1} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}, \vec{c_2} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \vec{c_3} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \left[\vec{x}\right]_B = \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}$$

$P_B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 6 & -1 \\ -2 & -5 & 4 \end{bmatrix}$ $P_C = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -3 & -1 \\ 7 & 5 & 2 \end{bmatrix}$

$P_B [\vec{x}]_B = P_C [\vec{x}]_C$

$[x]_B = \underbrace{P_B^{-1} P_C}_{P_{B \leftarrow C}} [x]_C$

$[\vec{x}]_C = \underbrace{P_C^{-1} P_B}_{P_{C \leftarrow B}} [\vec{x}]_B$

$\underset{B \leftarrow C}{P} = \begin{bmatrix} -29/6 & -1/3 & -1 \\ 2/3 & -1/3 & 0 \\ 1/6 & 2/3 & 0 \end{bmatrix}$

$\underset{C \leftarrow B}{P} = \begin{bmatrix} 0 & 4/3 & 2/3 \\ 0 & -1/3 & 4/3 \\ -1 & -19/3 & -11/3 \end{bmatrix}$

$\underset{C \leftarrow B}{P} \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}_B = \begin{bmatrix} 16/3 \\ 32/3 \\ -91/3 \end{bmatrix}_C = [\vec{x}]_C$