3/25/2021

Exponentials and Logarithms

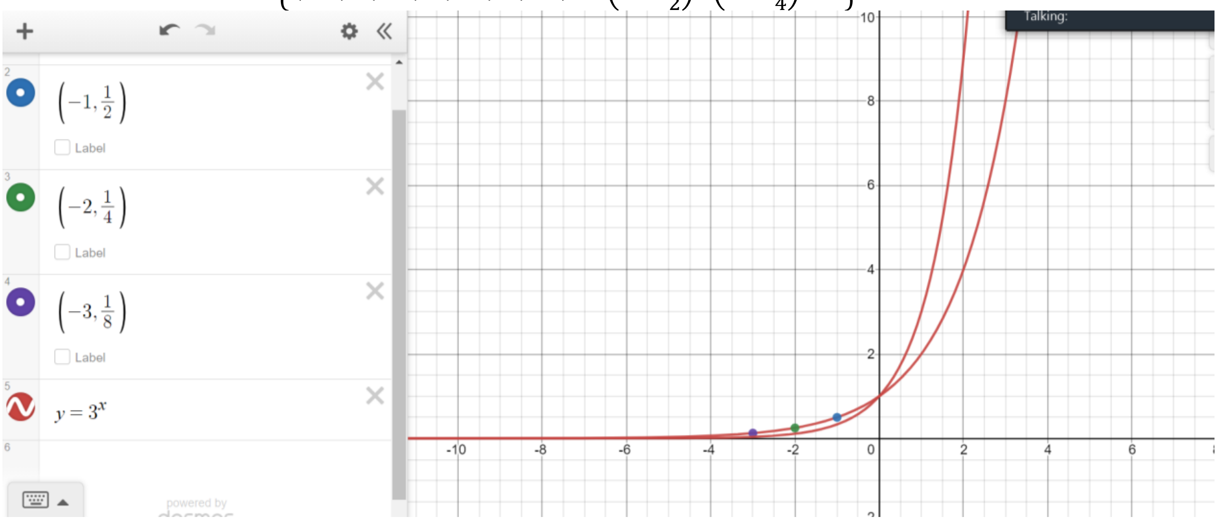
Exponential functions

$$f(x) = b^x$$

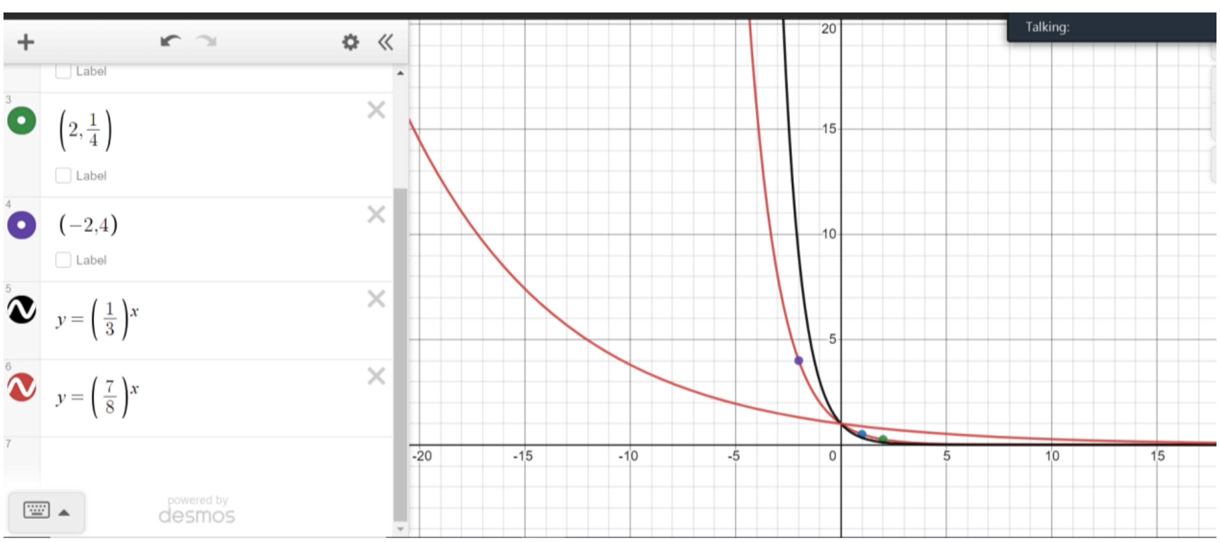
Base (b) cannot be 0, cannot be 1, cannot be negative

Case 1: $0 < b < 1$
Case 2: $b > 1$

Example of case 2: $f(x) = 2^x$
Points on this function $\left\{(0,1), (1,2), (2,4), (3,8) \ldots \left(-1, \frac{1}{2}\right), \left(-2, \frac{1}{4}\right), \ldots\right\}$

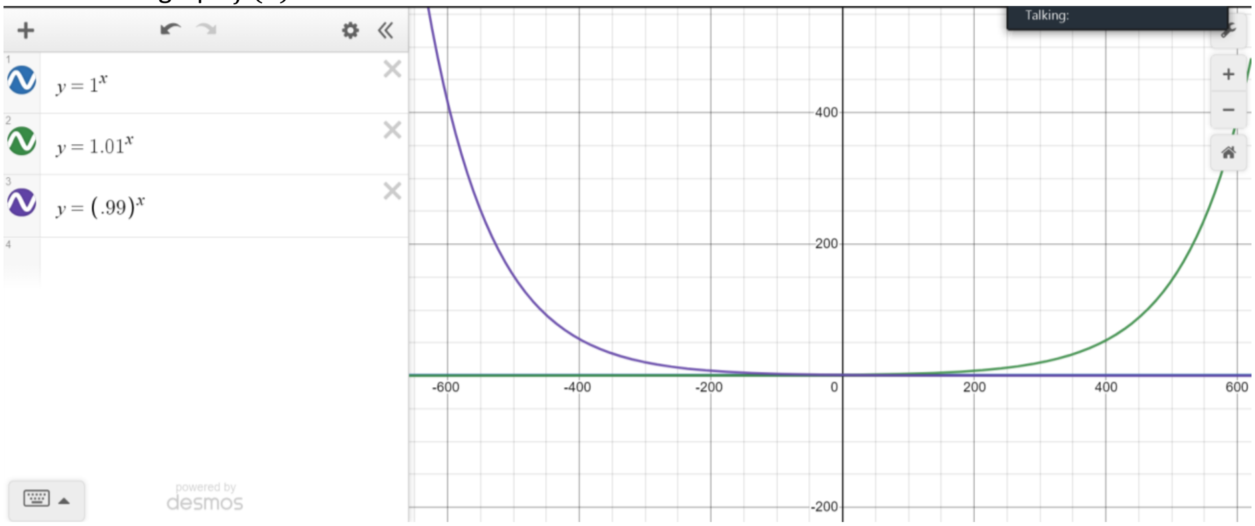


Example for case 1: $f(x) = \left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$



Domain and range:
Domain: all real numbers $(-\infty, \infty)$
Range: positive real numbers $(0, \infty)$

For the base graph $f(x) = b^x$



Properties of exponentials:

$$a^x \cdot b^x = (ab)^x$$
$$a^m \cdot a^n = a^{m+n}$$
$$\frac{a^m}{a^n} = a^{m-n}$$
$$a^{-n} = \frac{1}{a^n}$$

$$(a^m)^n = a^{mn}$$

Applying transformation to an exponential function.

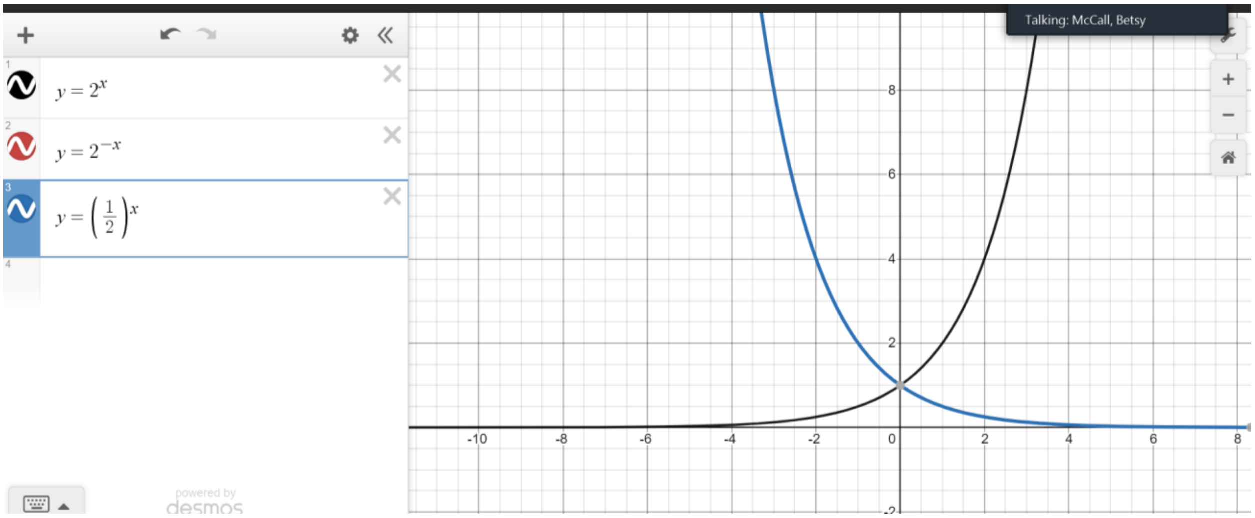
Common points on a standard base exponential graph:

$$\left\{\dots, \left(-1, \frac{1}{b}\right), (0,1), (1, b), \dots\right\}$$

Horizontal reflection: replace x with -x

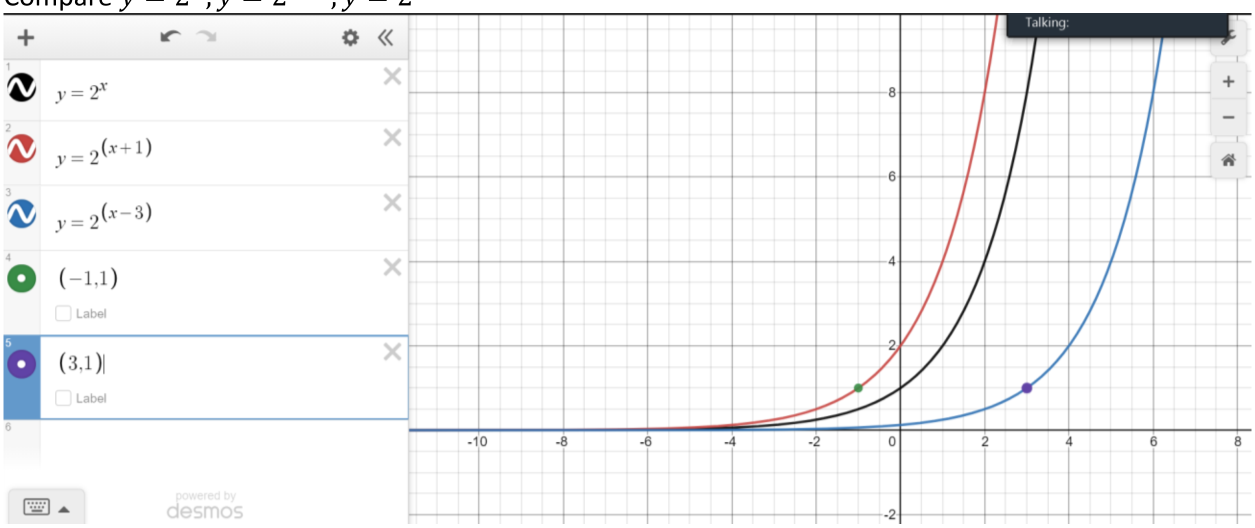
$$f(x) = 2^x$$
$$f(-x) = 2^{-x} = \left(\frac{1}{2}\right)^x$$

Equivalent to taking the reciprocal of the base.



Horizontal shift?

Compare $y = 2^x, y = 2^{x+1}, y = 2^{x-3}$



$$y = 2^{x-3} = \frac{2^x}{2^3} = \left(\frac{1}{8}\right)2^x$$
$$y = 2^{x+1} = 2^x(2^1) = 2(2^x)$$

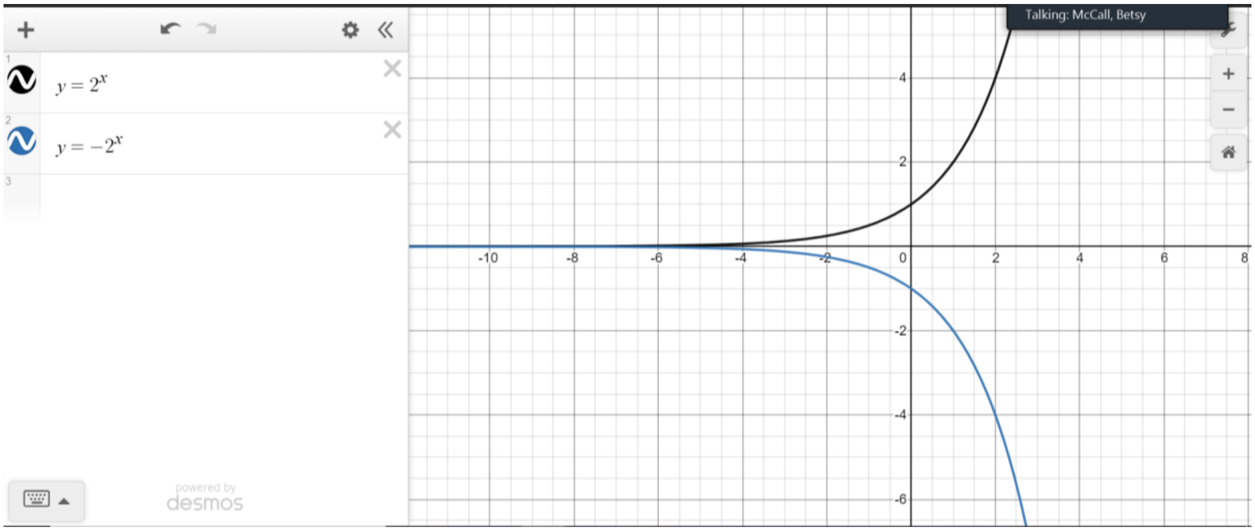
The horizontal shifts are equivalent to stretches and compressions.

$$y = 2^{3x} = (2^3)^x = 8^x$$

And $y = 2^{\frac{x}{3}} = \left(\sqrt[3]{2}\right)^x$

Vertical transformations:

Vertical reflection:
Multiply the function by (-1)… does not change the base

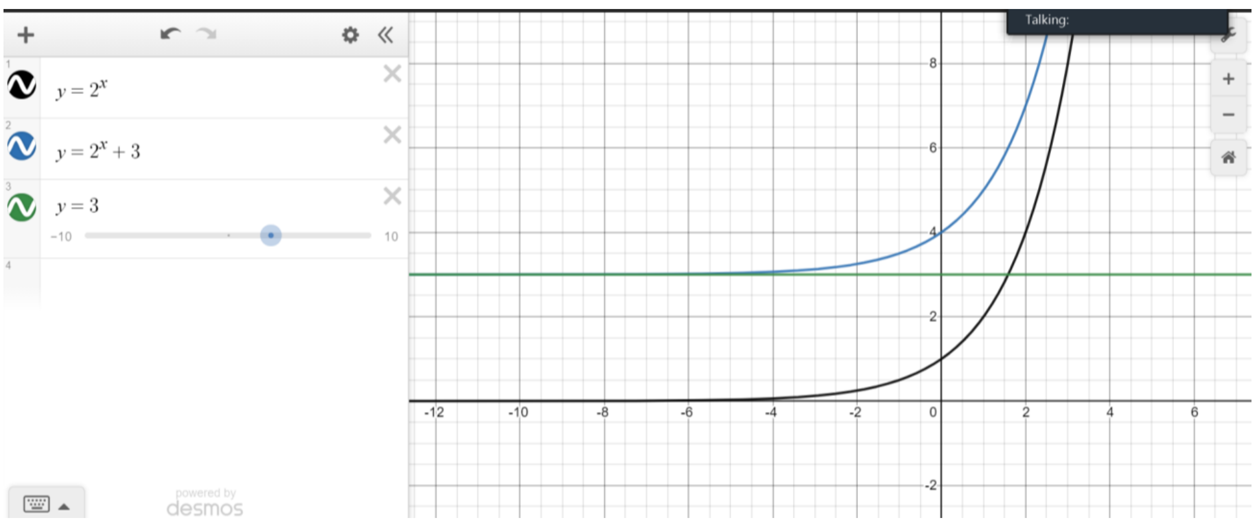


The range here is $(-\infty, 0)$

Vertical shifts move the available y-values up or down. Adjust the bottom value (non-infinite value) of the range.

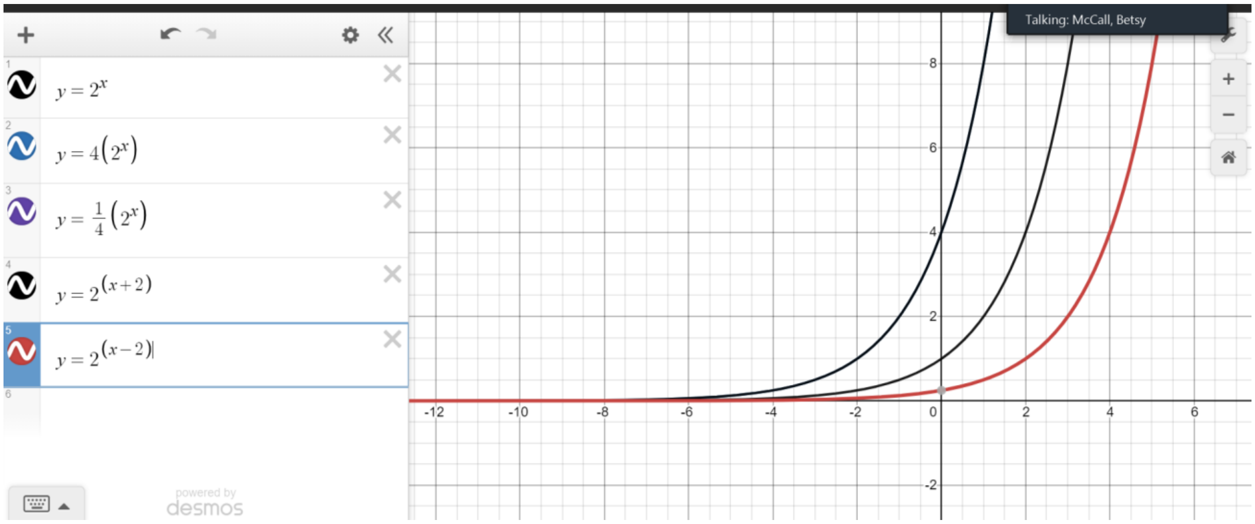
$$y = 2^x + 3$$

The range is $(3, \infty)$



Vertical stretch or compression:

$$y = 3(2^x)$$
$$y = \frac{1}{3}(2^x)$$



$$y=-2\left(\frac{2}{3}\right)^{x+1}$$

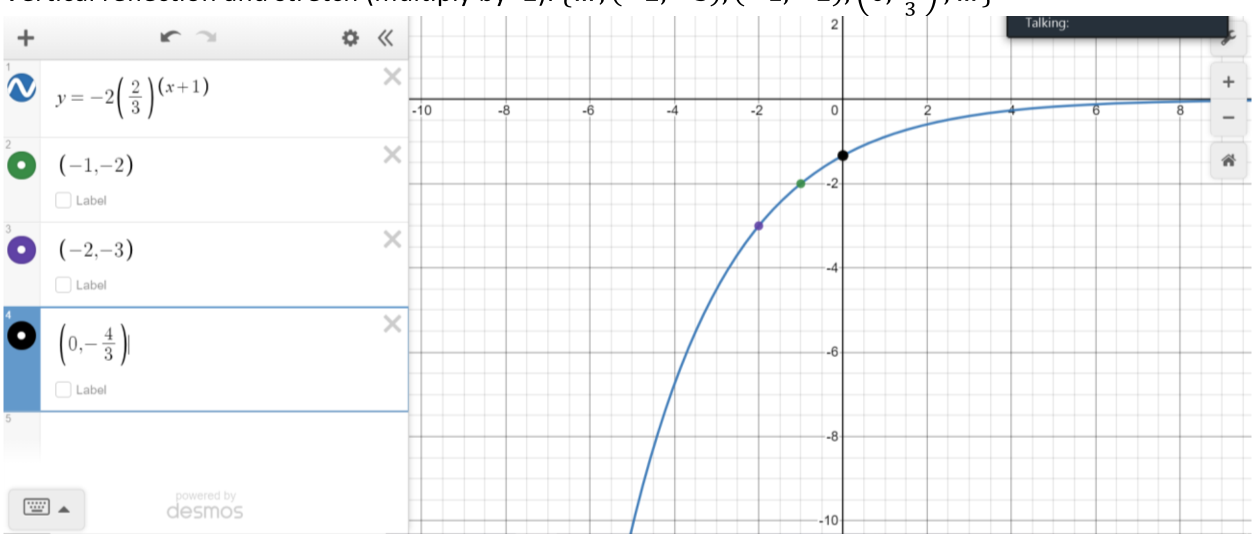
What is the asymptote? Only has a horizontal (no vertical) asymptote. No vertical shift: HA: y=0.

$$\{\ldots,\left(-1,\frac{1}{b}\right),(0,1),(1,b),\ldots\}$$
$$b=\frac{2}{3}$$

Base graph $y=\left(\frac{2}{3}\right)^x$: $\{\ldots,\left(-1,\frac{3}{2}\right),(0,1),\left(1,\frac{2}{3}\right),\ldots\}$

Horizontal shift left by 1: $\{\ldots,\left(-2,\frac{3}{2}\right),(-1,1),\left(0,\frac{2}{3}\right),\ldots\}$

Vertical reflection and stretch (multiply by -2): $\{\ldots,(-2,-3),(-1,-2),\left(0,\frac{-4}{3}\right),\ldots\}$



Natural exponential base $e\approx 2.71828\ldots$

$$f(x) = e^x$$

Logarithm definition

Essentially defined as the inverse function for exponentials.

$$2^x = 8$$

What number do I need to raise 2 to the power of in order to get 8?

$$2^x = 2^3$$

$$\log_2 8 = x = 3$$

$$\log_2 7 = ? \approx 2.81$$

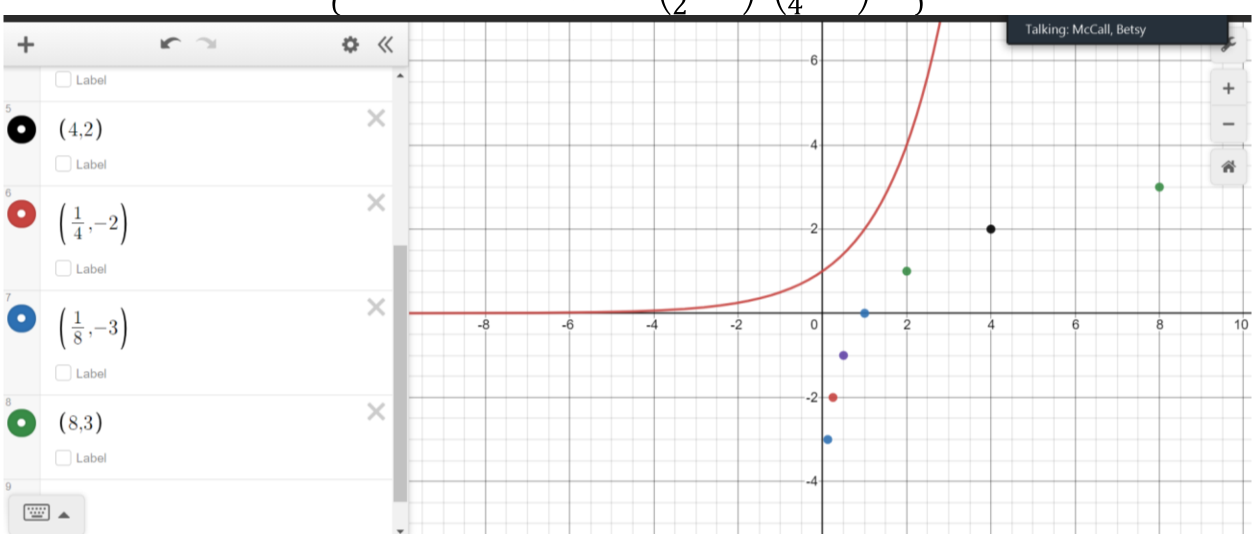
Logarithms are inverses of exponentials: So if we have pairs of points on an exponential curve, we can plot the graph of the logarithmic curve that is its inverse.

$$f(x) = 2^x$$

Points on this function $\left\{(0,1), (1,2), (2,4), (3,8) \ldots \left(-1, \frac{1}{2}\right), \left(-2, \frac{1}{4}\right), \ldots\right\}$

Points on the $g(x) = \log_2(x)$

$$\left\{(1,0), (2,1), (4,2), (8,3) \ldots \left(\frac{1}{2}, -1\right), \left(\frac{1}{4}, -2\right), \ldots\right\}$$



For logarithm functions (same base rules as exponentials)
The domain: $(0, \infty)$. The range is $(-\infty, \infty)$

The log functions have a vertical asymptote at x=0.
$\log x$ is assumed (in most cases) to be base-10 (in some textbooks, they may use log as the natural log)
$\ln(x)$ is assumed to be base e

Properties of logarithms

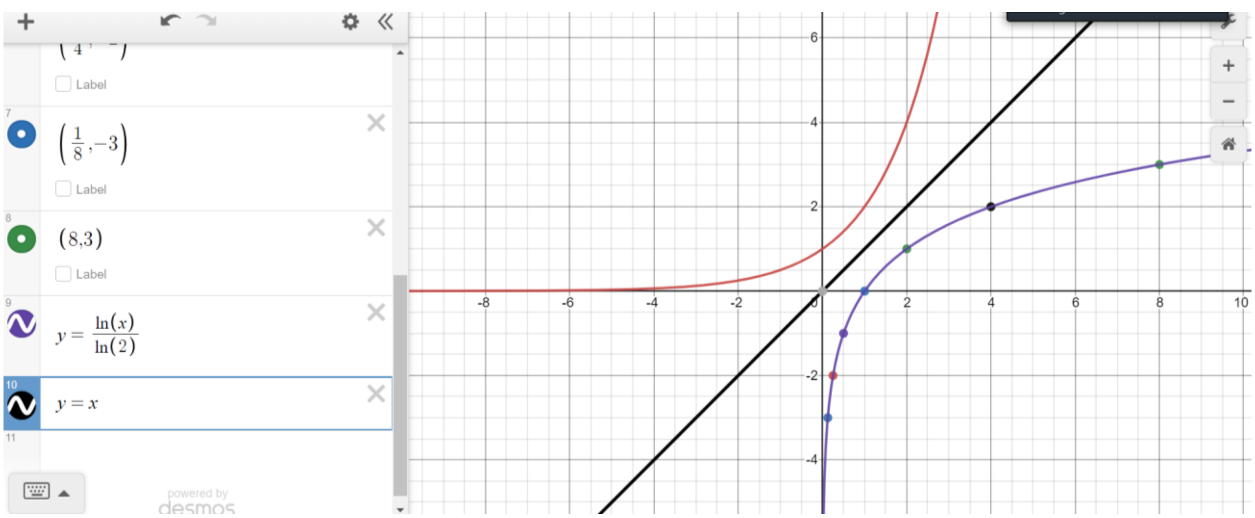
Exponential Properties	Logarithm Properties
$a^x \cdot b^x = (ab)^x$	
$a^m \cdot a^n = a^{m+n}$	$\log_a(MN) = \log_a M + \log_a N$
$\dfrac{a^m}{a^n} = a^{m-n}$	$\log_a\left(\dfrac{M}{N}\right) = \log_a M - \log_a N$
$a^{-n} = \dfrac{1}{a^n}$	
$(a^m)^n = a^{mn}$	$\log_a(M^n) = n \log_a M$

Change of base formula:

$$\log_a x = \frac{\ln x}{\ln a} = \frac{\log x}{\log a}$$

$$\log_2 81 = \frac{\ln 81}{\ln 2}$$

$$f(x) = \log_2(x) = \frac{\ln(x)}{\ln 2}$$



Expand the function into simple log expressions:

$$\log(x^2 + 2x + 1) = \log[(x + 1)^2] = 2\log\,(x + 1)$$

$$\log\left(\frac{x + 1}{x - 3}\right) = \log(x + 1) - \log(x - 3)$$

$$\log\big(4x(x-6)\big) = \log(4x) + \log(x-6) = \log(4) + \log(x) + \log(x-6)$$

$$\log\left(\sqrt{\frac{(x-1)(x-2)}{x+3}}\right) = \log\left(\left(\frac{(x-1)(x-2)}{x+3}\right)^{\frac{1}{2}}\right) = \frac{1}{2}\log\left(\frac{(x-1)(x-2)}{x+3}\right) =$$

$$\frac{1}{2}\big[\log\big((x-1)(x-2)\big) - \log(x+3)\big] = \frac{1}{2}\log(x-1) + \frac{1}{2}\log(x-2) - \frac{1}{2}\log(x+3)$$

Ex. For combining

$$\log(x+1) + \log(x-3) = \log 15$$

Can I get to the point where log(M) = log(N)?? if so, we can cancel logs.

$$\log[(x+1)(x-3)] = \log(15)$$

I can cancel logs now

$$(x+1)(x-3) = 15$$

I can solve for x from here.

$$2^x = 8$$
$$2^x = 2^3$$
$$x = 3$$