MTH 263, Final Exam, Spring 2022 Name KEY

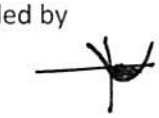
Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error or any kind.

1. Find the area of the region bounded by $y = x^3, x = 0, y = 0, x = 2$. Round your answer to 4 decimal places if needed. (12 points)

$$\int_0^2 x^3\,dx = \frac{1}{4}x^4\Big|_0^2 = \frac{1}{4}(16) = \boxed{4}$$

2. Find the center of mass of the lamina with constant density of the region bounded by $y = x^2 - x, x = 0$. (16 points) $x(x-1)=0,\ x=0,\ x=1$



$$M = \rho\int_0^1 x^2 - x\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2\Big|_0^1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6} \rightarrow \text{must be positive} \quad M = \frac{1}{6}$$

$$\text{or } \int_0^1 x - x^2\,dx$$

$$M_x = \int_0^1 (x^2-x)^2\,dx = \int_0^1 x^4 - 2x^3 + x^2\,dx = \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3\Big|_0^1 = \frac{1}{5} - \frac{1}{2} + \frac{1}{3} = \frac{1}{30}$$

$$M_y = \int_0^1 x(x^2-x)\,dx = \int_0^1 x^3 - x^2\,dx = \frac{1}{4}x^4 - \frac{1}{3}x^3\Big|_0^1 = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$$

$$\bar{x} = \frac{-1/12}{-1/6} = \frac{6}{12} = \frac{1}{2}$$

$$\bar{y} = \frac{1/30}{-1/6} = -\frac{6}{30} = -\frac{1}{5}$$

$$\left(\frac{1}{2}, -\frac{1}{5}\right)$$

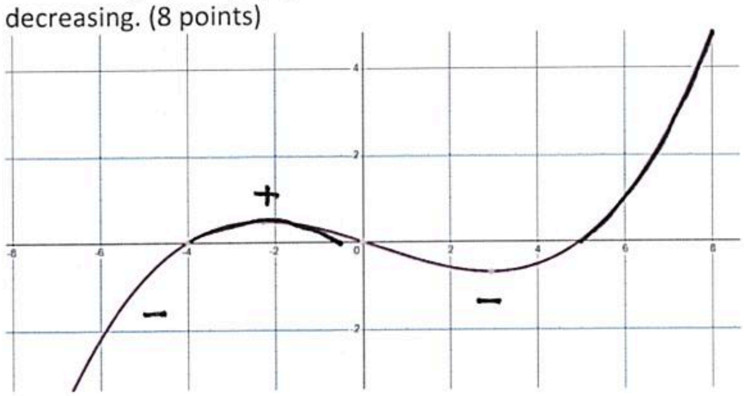
3. Find the length of the arc of the curve $f(x) = \cosh x$ on the interval [0,2]. Round to 4 decimal places. (8 points)

$f'(x) = \sinh x$

$$\int_0^2 \sqrt{1+\sinh^2 x}\, dx = \int_0^2 \sqrt{\cosh^2 x}\, dx = \int_0^2 \cosh x \, dx =$$

$$\sinh x \Big|_0^2 = \sinh 2 \approx 3.62686\ldots$$

4. Given the graph of $f'(x)$ shown below, determine the intervals on which $f(x)$ is increasing or decreasing. (8 points)



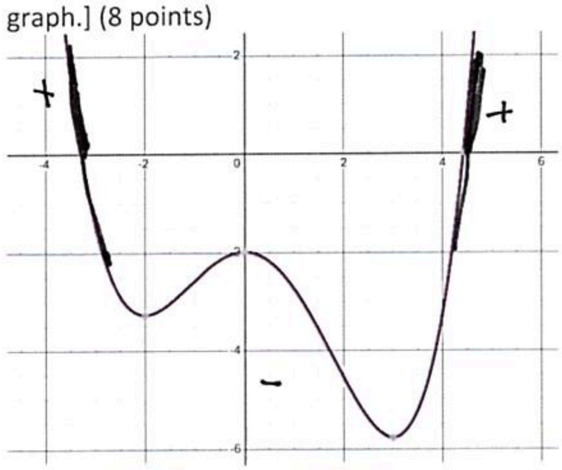
increasing

$(-4,0) \cup (5,\infty)$

decreasing

$(-\infty,-4) \cup (0,5)$

5. Given the graph of $f'(x)$, determine the intervals on which $f(x)$ is concave up or concave down. [Hint: first sketch the second derivative based on the first, then find the intervals from that graph.] (8 points)



Concave up

$(-2,0) \cup (3,\infty)$

Concave down

$(-\infty,-2) \cup (0,3)$

6. Use differentials to estimate the value of $\sqrt[4]{81.1}$. Round your answer to ~~9~~ 7 decimal places if needed. (10 points)

$f(x) = x^{1/4}$ $\quad f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4\sqrt[4]{x^3}}$ $\quad x = 81$

$f(81) = 3$ $\quad f'(81) = \frac{1}{4 \cdot 27} = \frac{1}{108}$ $\quad \Delta x = 0.1$

$3 + \frac{1}{108}(0.1) = 3.000925926$

exact 3.000925498

7. Find the absolute extrema of $f(x) = 3x^4 - 4x^3 - 12x^2 + 6$ on the interval [-1,5]. (12 points)

$f'(x) = 12x^3 - 12x^2 - 24x = 12(x)(x^2 - x - 2) = 12x(x-2)(x+1)$

$x = 0, 2, -1$

all pts on interval

$f(-1) = 1$ $\qquad f(5) = 1081 \leftarrow$ abs. maximum

$f(2) = -26 \rightarrow$ abs. minimum

$f(0) = 6$

8. Find the value of each limit. (8 points each for a-c)

a. $\lim\limits_{x\to\infty} \sin\left(\frac{1}{x}\right) = 0$

b. $\lim\limits_{x\to\infty} (2x)^{1/x} = e^0 = 1$

$e^{\lim_{x\to\infty} \ln(2x)^{1/x}} \quad \lim\limits_{x\to\infty} \frac{1}{x}\ln(2x) = \lim\limits_{x\to\infty} \frac{\ln 2x}{x} = \lim\limits_{x\to\infty} \frac{\frac{1}{2x}\cdot 2}{1} = 0$

c. $\lim\limits_{x\to 0} \frac{\cos(3x) - \cos(11x)}{x^2} = \frac{11\cancel{2} - 9}{2} = \frac{121 - 9}{2} = 56$

$\lim\limits_{x\to 0} \frac{-3\sin 3x + 11\sin 11x}{2x} = \lim\limits_{x\to 0} \frac{-9\cos 3x + 121\cos 11x}{2} = \frac{-9+121}{2} = \frac{112}{2} = 56$

d. The following questions refer to the graph of $f(x)$ shown. (3 points each)

i. $\lim_{x\to 0^+} f(x)$ $= 4$

ii. $\lim_{x\to 0^-} f(x)$ $= 0$

iii. $\lim_{x\to 0} f(x)$ = Does not exist

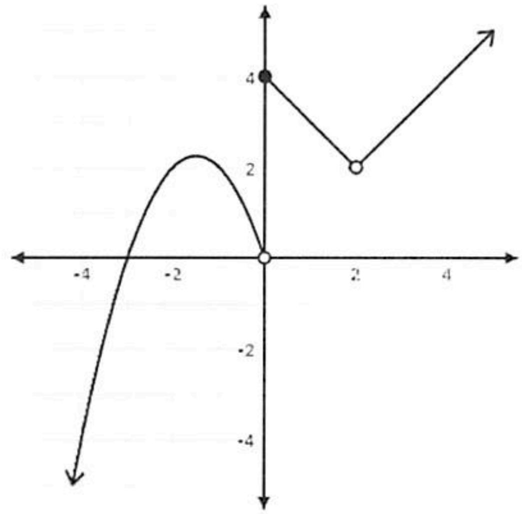
iv. $\lim_{x\to 2^-} f(x)$ $= 2$

v. $\lim_{x\to 2^+} f(x)$ $= 2$

vi. $\lim_{x\to 2} f(x)$ $: 2$

vii. Is the graph continuous at $x = 0$? no

viii. Is the graph continuous at $x = 2$? no (f(2) does not exist, not defined)



9. Find the value of the indicated derivative. (6 points each)

a. $f(x) = 2x^3 - \frac{12}{x^2}, f'(-2)$ $\quad 2x^3 - 12x^{-2}$

$$f'(x) = 6x^2 + \frac{24}{x^3} \qquad f'(-2) = 6(-2)^2 + \frac{24}{(-2)^3} = 24 - 3 = \boxed{21}$$

b. $f(x) = (9 - x^2)^3, f''(1)$

$$f'(x) = 3(9-x^2)^2(-2x) = -6x(9-x^2)^2$$

$$f''(x) = -6(9-x^2)^2 - 6x(9-x^2)2(-2x) = -6(8)^2 - 6(1)(8)(2)(-2) = \boxed{-192}$$

$$-6(9-x^2)^2 + 24x^2(9-x^2) = -6(8)^2 + 24(8) =$$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

10. Find the volume of the solid of revolution for the region bounded by $x = y^2$ and $x = 3y$ rotated around the y-axis. (13 points)

$$y^2 = 3y \rightarrow y^2 - 3y = 0 \rightarrow y(y-3) = 0$$

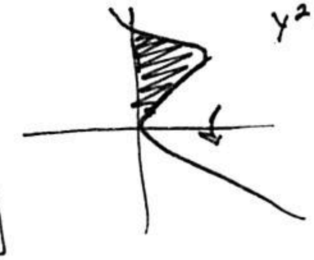
$$y = 0,\ y = 3$$

$$-\pi\int_0^3 (y^2)^2 - (3y)^2\,dy = -\pi\int_0^3 y^4 - 9y^2\,dy =$$

$$-\pi\left[\tfrac{1}{5}y^5 - 3y^3\right|_0^3 = -\pi\left[\tfrac{1}{5}\cdot 243 - 81\right] = \pi\left[\tfrac{162}{5}\right] = \boxed{\frac{162\pi}{5}}$$

11. Find the volume of the solid of revolution for the region bounded by $x = 2y^2 - y^3, x = 0, y = 0$ rotated around the x-axis. (12 points)

$y^2(2-y)$



$$2\pi\int_0^2 y(2y^2-y^3)\,dy = 2\pi\int_0^2 2y^3-y^4\,dy$$

$$= 2\pi\left[\tfrac{1}{2}y^4 - \tfrac{1}{5}y^5\right]_0^2 = 2\pi\left[8-\tfrac{32}{5}\right] = \boxed{16\pi/5}$$

12. Find the surface area of the surface defined by $y = \frac{1}{x}$, on the interval [1,4], revolved around the y-axis. (12 points)

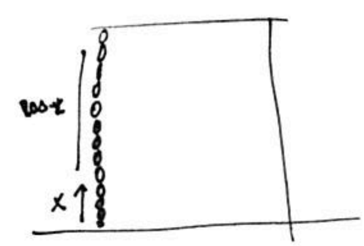
$$2\pi\int_1^4 x\sqrt{1+\left(\tfrac{1}{x^2}\right)^2}\,dx$$

$$=2\pi\int_1^4 x\sqrt{\frac{x^4+1}{x^4}}\,dx = 2\pi\int_1^4 \frac{\cancel{x}\sqrt{x^4+1}}{\cancel{x^2}}\,dx = 2\pi\int_1^4 \frac{\sqrt{x^4+1}}{x}\,dx$$

$$2\pi(7.71793\ldots)$$

≈ 48.49 integrated numerically

13. Find the work done by winding up a hanging cable of length 100 feet, and weight density 5 lbs/ft. (15 points)



$$\int_0^{100} 5\,(100-x)\,dx = \int_0^{100} 500 - 5x\,dx =$$

$$500x - \tfrac{5}{2}x^2\Big|_0^{100} = 50{,}000 - \tfrac{5}{2}(10{,}000) =$$

$$25{,}000 \text{ J}$$

14. Create a sign chart for the first and second derivatives of $f(x) = x^{17/3} - x^{11/5}$. Identify any critical points, cusps, and inflection points. Use this information to sketch a graph of the curve. (10 points)

$$f'(x) = \tfrac{17}{3} x^{14/3} - \tfrac{11}{5} x^{6/5}$$

$$\tfrac{17}{3} x^{70/15} - \tfrac{11}{5} x^{18/15} = x^{18/15}\left(\tfrac{17}{3} x^{52/15} - \tfrac{11}{5}\right)$$

$x = 0$

$\frac{17}{3} x^{52/15} - \frac{11}{5} = 0 \qquad \frac{17}{3} x^{52/15} = \frac{11}{5}$

$x^{52/15} = \frac{11}{5} \cdot \frac{3}{17} = \frac{33}{85}$

$$x = \sqrt[52]{\left(\tfrac{33}{85}\right)^{15}} \approx \pm .761149$$

$$f''(x) = \tfrac{17}{3} \cdot \tfrac{14}{3} x^{11/3} - \tfrac{11}{5} \cdot \tfrac{6}{5} x^{1/5} = \tfrac{238}{9} x^{11/3} - \tfrac{66}{25} x^{1/5}$$

$$\tfrac{238}{9} x^{55/15} - \tfrac{66}{25} x^{3/15}$$

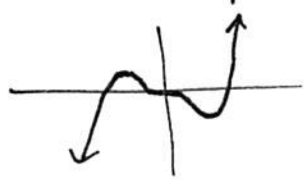
$$x^{3/15}\left(\tfrac{238}{9} x^{52/15} - \tfrac{66}{25}\right) = 0 \qquad x = 0$$

$$x = \pm\left(\tfrac{66}{25} \cdot \tfrac{9}{238}\right)^{15/52}$$

$$x \approx \pm 0.5144317$$

f': -0.76 $\quad 0 \quad$ 0.76

f'': -0.51 $\quad 0 \quad$ 0.51



15. Find the antiderivatives. (8 points each)

a. $\int x^{\pi} + 4\sec(2x)\tan(2x) - 8 dx$

$$\frac{1}{\pi+1} x^{\pi+1} + 2\sec 2x - 8x + C$$

b. $\int e^x \cos(e^x) - \operatorname{csch}^2 x - \frac{1}{\sqrt{1-x^2}} dx$

$$\sin e^x + \coth x - \arcsin x + C$$

c. $\int \frac{\ln x}{x} - \frac{1}{x-2} - 2^x dx$

$u = \ln x \quad du = \frac{1}{x} dx$

$$\frac{(\ln x)^2}{2} - \ln|x-2| - \frac{2^x}{\ln 2} + C$$

d. $\int \frac{x}{1+x^4}dx$

$u = x^2 \quad du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$\frac{1}{2}\arctan x^2 + C$

16. Find the limit: $\lim_{x\to 25} \frac{x-25}{\sqrt{x}-5}$ (8 points)

$\lim_{x\to 25} \frac{(x-25)(\sqrt{x}+5)}{(\sqrt{x}-5)(\sqrt{x}+5)} = \lim_{x\to 25} \frac{\cancel{(x-25)}(\sqrt{x}+5)}{\cancel{x-25}} =$

$\lim_{x\to 25} \sqrt{x}+5 = \boxed{10}$

17. A particle is moving along a trajectory defined by $s(t) = t^3 - 3t + 7$. Find the instantaneous velocity at the point $\boldsymbol{t} = 2$ using the limit definition of the derivative. You should provide both the general equation, and the velocity at that point. (8 points)

$s'(t) = 3t^2 - 3$

$s'(2) = 12 - 3 = 9$

$s'(t) = \lim_{h\to 0} \frac{(t+h)^3 - 3(t+h) + 7 - (t^3 - 3t + 7)}{h} =$

$\lim_{h\to 0} \frac{\cancel{t^3} + 3th^2 + 3t^2h + h^3 - \cancel{3t} - 3h + \cancel{7} - \cancel{t^3} + \cancel{3t} - \cancel{7}}{h}$

$\lim_{h\to 0} \frac{3th^2 + 3t^2h + h^3 - 3h}{h} = \lim_{h\to 0} \frac{\cancel{h}(3th + 3t^2 + h^2 - 3)}{\cancel{h}} =$

$\lim_{h\to 0} 3th + 3t^2 + h^2 - 3 = 3t^2 - 3$

18. Find the derivative of each of the following. If no derivative notation is specified, find the first derivative only. If a derivative notation is specified with the function, find the indicated derivative. (8 points each)

a. $f(x) = e^{\csc x}$

$f'(x) = -e^{\csc x} \csc x \cot x$

b. $f(x) = x \sin x, f''(x)$

$$f'(x) = \sin x + x\cos x$$

$$f''(x) = \cos x + \cos x + x(-\sin x) = 2\cos x - x\sin x$$

c. $f(x) = \arcsin(e^{x^2})$

$$f'(x) = \frac{1}{\sqrt{1-(e^{x^2})^2}} \cdot e^{x^2} \cdot 2x = \frac{2xe^{x^2}}{\sqrt{1-e^{2x^2}}}$$

d. $xy^2 + y = xy - \ln x, \frac{dy}{dx}$

$$y^2 + x \cdot 2yy' + y' = y + xy' - \frac{1}{x}$$

$$2xyy' + y' - xy' = y - y^2 - \frac{1}{x}$$

$$y'(2xy + 1 - x) = y - y^2 - \frac{1}{x}$$

$$\frac{dy}{dx} = y' = \frac{y - y^2 - \frac{1}{x}}{2xy + 1 - x} \quad \text{or} \quad y' = \frac{xy - xy^2 - 1}{2x^2y + x - x^2}$$