2/10/2022

3.6 Chain Rule (continued)
3.7 Derivatives of inverse functions

Generalized Power Rule

$$h(x) = \left(g(x)\right)^n$$
$$h'(x) = n\left(g(x)\right)^{n-1} g'(x)$$

Suppose $h(x) = (2x^3 + 5x - 7)^4$. Find $h'(x)$.

Think of this as $(stuff)^4$

$$h'(x) = 4(stuff)^3 \times [stuff]'$$
$$= 4(2x^3 + 5x - 7)^3(6x^2 + 5)$$
$$= (24x^2 + 20)(2x^3 + 5x - 7)^3$$

Combining the product rule and the chain rule.

$$h(x) = (6x^2 + 11)^3(2x^4 + 5x)$$

$f(x) = (6x^2 + 11)^3$	$g(x) = 2x^4 + 5x$
$f'(x) = 3(6x^2 + 11)^2(12x) = 36x(6x^2 + 11)^2$	$g'(x) = 8x^3 + 5$

$$h'(x) = 36x(6x^2 + 11)^2(2x^4 + 5x) + (6x^2 + 11)^3(8x^3 + 5)$$

$$h(x) = [x \sin x]^2 = x^2 \sin^2 x$$

$$h(x) = [x \sin x + x - 2]^5$$

$$h'(x) = 5[x \sin x + x - 2]^4 \times [x \sin x + x - 2]'$$

$f(x) = x$	$g(x) = \sin x$
$f'(x) = 1$	$g'(x) = \cos x$

$$[x \sin x]' = \sin x + x \cos x$$
$$h'(x) = 5[x \sin x + x - 2]^4[\sin x + x \cos x + 1]$$

Quotient Rule
Derive the quotient rule from a combination of the chain rule and the product rule

$$h(x) = \frac{f(x)}{g(x)} = f(x)[g(x)]^{-1}$$

$$h(x) = p(x)q(x), h'(x) = p'(x)q(x) + p(x)q'(x)$$

$p(x) = f(x)$	$q(x) = [g(x)]^{-1}$
$p'(x) = f'(x)$	$q'(x) = -1[g(x)]^{-2}g'(x)$

$$h'(x) = f'(x)[g(x)]^{-1} + f(x)(-[g(x)]^{-2})g'(x)$$
$$= f'(x)[g(x)]^{-1} - f(x)[g(x)]^{-2}g'(x)$$
$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{\left(g(x)\right)^2} == \frac{f'(x)}{g(x)} \cdot \frac{g(x)}{g(x)} - \frac{f(x)g'(x)}{\left(g(x)\right)^2} = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

Example.

$$h(x) = \frac{4x+2}{5x-1} = (4x+2)(5x-1)^{-1}$$

$f(x) = 4x + 2$	$g(x) = (5x-1)^{-1}$
$f'(x) = 4$	$g'(x) = -1(5x-1)^{-2}(5)$

$$h'(x) = (4x+2)(-5(5x-1)^{-2}) + (4)(5x-1)^{-1}$$

$$\frac{(-5)(4x+2)}{(5x-1)^2} + \frac{4}{5x-1}\frac{(5x-1)}{(5x-1)} = \frac{4(5x-1) - 5(4x+2)}{(5x-1)^2}$$

Using quotient rule

$f(x) = 4x + 2$	$g(x) = 5x - 1$
$f'(x) = 4$	$g'(x) = 5$

$$h'(x) = \frac{4(5x-1) - 5(4x+2)}{(5x-1)^2}$$

$$F(x) = \cot^3(4x+1)$$
$$F(x) = [\cot(4x+1)]^3$$

$$F'(x) = 3(stuff)^2(stuff)'$$
$$= 3(\cot(4x+1))^2(-\csc^2(4x+1))(more\ stuff)'$$
$$= 3(\cot(4x+1))^2(-\csc^2(4x+1))(4)$$
$$3\cot^2(4x+1)\,(-\csc^2(4x+1))(4)$$

$$h(x) = \sin(\cos 7x)$$

$$h'(x) = \cos(\cos 7x)\,(-\sin(7x))(7)$$

$$h(x) = \csc^2(x^3 + \sin(8x))$$

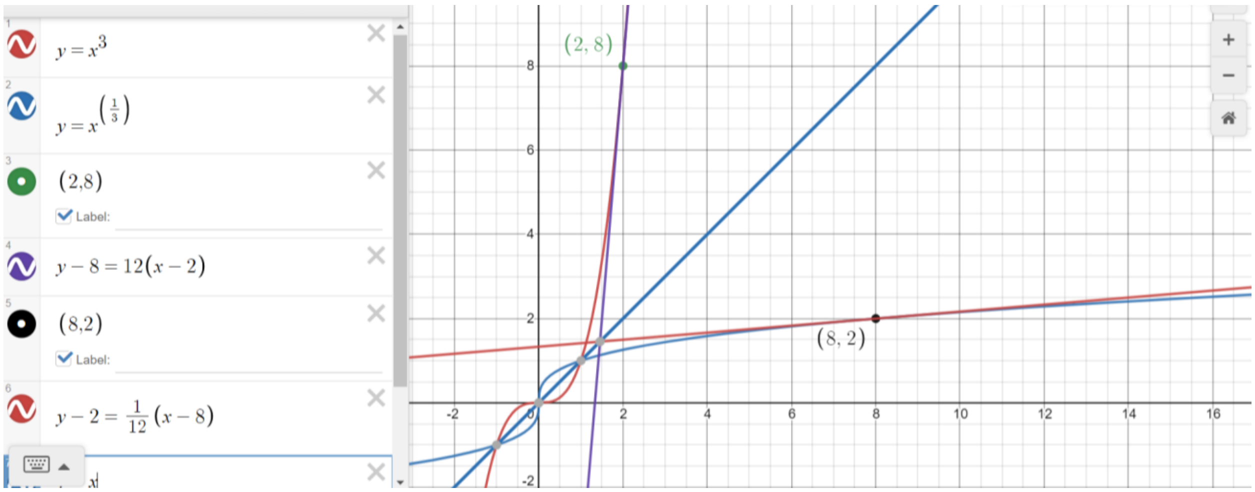
$$h'(x) = 2\csc(x^3 + \sin(8x))\,(\csc(x^3 + \sin(8x)))'$$
$$= 2\csc(x^3 + \sin(8x))\,(-\csc(x^3 + \sin(8x))\cot(x^3 + \sin(8x)))(x^3 + \sin(8x))'$$
$$= 2\csc(x^3 + \sin(8x))\,(-\csc(x^3 + \sin(8x))\cot(x^3 + \sin(8x)))(3x^2 + \cos(8x)(8))$$

$$-2\csc^2(x^3 + \sin(8x))\cot(x^3 + \sin(8x))\,(3x^2 + 8\cos(8x))$$

3.7 Derivatives of inverse functions

Suppose $y = f^{-1}(x)$. Think of x as $f^{-1}\left(f(x)\right) = x = f\left(f^{-1}(x)\right)$

$$\frac{dy}{dx} = \frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$



The slopes of the corresponding points on the function and its derivative are reciprocals of each other.

Derivatives for inverse trig functions

This rule can also be used for proving that the power rule applies to root/radical functions.

$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}[\cos^{-1}(x)] = \frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}[\tan^{-1} x] = \frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$
$$\frac{d}{dx}[\cot^{-1} x] = \frac{d}{dx}[\operatorname{arccot} x] = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}[\sec^{-1} x] = \frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}[\csc^{-1} x] = \frac{d}{dx}[\operatorname{arccsc} x] = -\frac{1}{|x|\sqrt{x^2-1}}$$

If absolute values are left off x in these last two cases, it's because they are assuming x is positive.

Find $f'(x)$ if $f(x) = \arctan(x^2)$

$$f'(x) = \frac{1}{1+(stuff)^2}(stuff)' = \frac{1}{1+(x^2)^2}(2x) = \frac{2x}{1+x^4}$$

Find $h'(x)$ is $h(x) = x^3 \sin^{-1} x$

$f(x) = x^3$	$g(x) = \sin^{-1} x$
$f'(x) = 3x^2$	$g'(x) = \frac{1}{\sqrt{1-x^2}}$

$$h'(x) = 3x^2 \sin^{-1} x + \frac{x^3}{\sqrt{1 - x^2}}$$