119
100

MTH 264, Exam #2, Spring 2024 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. For each of the series below, determine whether the series converges or diverges (in #7, you'll be asked to prove your conclusion, so it may help to do those problems first/together). (~~5~~ 6 points each)
   a. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ Converges, alternating series test
   b. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges, integral test
   c. $\sum_{n=0}^{\infty} \frac{2^n-1}{7^n+1}$ converges, limit comparison test w/ $\left(\frac{2}{7}\right)^n$ ~ by ratio test
   d. $\sum_{n=2}^{\infty} \frac{\ln n}{n^4}$ Converges, by direct comparison test w/ $\frac{1}{n^3}$ (p-series)
   e. $\sum_{n=2}^{\infty} \frac{\arccos n}{\sqrt{1-n^2}}$ diverges, by the integral test

   (I had changed the problem, but it didn't change the result)
   f. $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$ Converges, telescoping series test

g. $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$ converges, by ratio test

h. $\sum_{n=1}^{\infty} \left(\frac{8n}{3n-2}\right)^n$ diverges, by root test

2. Find N such that $R_N \leq 10^{-5}$, for the convergent series. (10 points)

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$$

$$\frac{2^{4n}}{(2n+1)!} \leq 10^{-5}$$

$$2^{4n} \cdot 10^5 \leq (2n+1)!$$

$n = 9$ (error)

need 8 terms

3. Find the fourth Taylor polynomial for the function $f(x) = \sqrt{x}$ around the point c = 2. What is the maximum error for P_4 found above at x=2.1? What is the actual error? (9 points) $f(2) = \sqrt{2}$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(2) = \frac{1}{2\sqrt{2}}$

$f''(x) = \frac{-1}{4}x^{-3/2}$ $\quad f''(2) = \frac{-1}{4\sqrt{8}} = \frac{-1}{8\sqrt{2}}$

$f'''(x) = \frac{3}{8}x^{-5/2}$ $\quad f'''(2) = \frac{+3}{8\sqrt{32}} = \frac{3}{32\sqrt{2}}$

$f^{iv}(x) = -\frac{15}{16}x^{-7/2}$ $\quad f^{iv}(2) = \frac{-15}{16\sqrt{128}} = \frac{-15}{128\sqrt{2}}$

$f^{v}(x) = \frac{105}{32}(x)^{-9/2}$ $\quad f^{v}(2) = \frac{105}{32\sqrt{512}} = \frac{105}{512\sqrt{2}}$

$\frac{105}{512\sqrt{2}}(1/120)(2.1-2)^5$

$P_4 = \sqrt{2} + \frac{1}{2\sqrt{2}}(x-2) - \frac{1}{16\sqrt{2}}(x-2)^2 + \frac{3}{96\sqrt{2}}(x-2)^3 - \frac{15}{3072\sqrt{2}}(x-2)^4$

$P_4(2.1) \approx 1.449148712$

$\sqrt{2.1} \approx 1.449137675$

true error $\approx 1.100368952 \times 10^{-5}$

4. List the first five terms of the sequences below. ($\cancel{6}$ 12 points each)

a. $a_n = \frac{(-2)^{n+1}}{2^n+1}$ $\quad a_0 = \frac{-2}{2} = -1$, $a_1 = \frac{4}{3}$, $a_2 = \frac{-8}{5}$, $a_3 = \frac{16}{9}$, $a_4 = \frac{-32}{17}$, $a_5 = \frac{64}{33}$

b. $b_{n+1} = 2b_n^2 - n + 1, b_0 = 1$

$b_0 = 1$, $b_1 = 2-1 = 1$, $b_2 = 2-2 = 0$, $b_3 = -3$, $b_4 = 2(9)-4 = \cancel{16}$ 14, $b_5 = \cancel{512}-5 = \cancel{507}$ 387

5. Find the limit of the sequence if it exists. If it does not, state that it diverges. (6 points each)

a. $a_n = \frac{1}{2}\arctan(n)$ $\lim_{n\to\infty} \frac{1}{2}\arctan n = \pi/4$

b. $b_n = \frac{(-1)^{n+1}n}{\sqrt{n^5+3}}$ $\lim_{n\to\infty} \frac{(-1)^{n+1}n}{\sqrt{n^5+3}} = 0$

c. $c_n = \ln(\sin\frac{1}{n}) + \ln(n)$ $\lim_{n\to\infty} \ln(\sin\frac{1}{n}) + \ln(n) = 0$

$\lim_{n\to\infty} \ln(\sin\frac{1}{n}\cdot n) = \lim_{n\to\infty} \ln \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \ln(1) = 0$

6. Find the interval of convergence of the power series. (10 points each)

$$\sum_{n=0}^{\infty} \frac{(n!)^2 x^n}{(2n)!}$$

$\lim_{n\to\infty} \left| \frac{(n+1)!^2 x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!^2 x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 x}{(2n+1)(2n+2)} \right| = \lim_{n\to\infty} \left| \frac{x(n+1)}{(2n+1)2} \right| = \frac{x}{4} < 1$

$-1 < \frac{x}{4} < 1 \rightarrow$

$-4 < x < 4$

$\sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n)!}$ diverges $\sum_{n=0}^{\infty} \frac{(n!)^2 (-4)^n}{(2n)!}$ also diverges $(-4, 4)$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

7. For each of the series below (same as in #1), state the name of the test used to determine convergence. Show the work here to support your conclusion above. (8 points each)

a. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ converges, by alternating series test

b. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ $\int_2^{\infty} \frac{1}{x \ln x} dx$ $u = \ln x$, $du = \frac{1}{x} dx$ $\int \frac{1}{u} du \rightarrow \ln(\ln x)) \Big|_2^{\infty} = \infty$

diverges by integral test.

c. $\sum_{n=0}^{\infty} \frac{2^n - 1}{7^n + 1}$ $\lim_{n\to\infty} \frac{2^n - 1}{7^n + 1} \cdot \frac{7^n}{2^n} = \lim_{n\to\infty} \frac{1 - 1/2^n}{1 + 1/7^n} = 1$ $\frac{2^n}{7^n}$ converges by the ratio test or geometric series test

converges by limit comparison

d. $\sum_{n=2}^{\infty} \frac{\ln n}{n^4}$ $\ln n < n$ $\therefore \frac{\ln n}{n^4} < \frac{n}{n^4} = \frac{1}{n^3}$ $\frac{1}{n^3}$ converges by p-series test

$\therefore$ Converges by direct comparison

e. $\sum_{n=2}^{\infty} \frac{\arccos n}{\sqrt{1-n^2}}$ $\int_2^{\infty} \frac{\arccos x}{\sqrt{1-x^2}}\,dx$ $-\frac{1}{2}(\arccos x)^2 \Big|_2^{\infty} =$ diverges

by integral test

f. $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$ $\lim_{n\to\infty} \frac{1}{n+2} = 0$ converges to $1/2$ by telescoping series test

g. $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$ $\lim_{n\to\infty} \left| \frac{(-2)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-2)^n} \right| = \lim_{n\to\infty} \left| \frac{-2}{n+1} \right| = 0 < 1$

converges by ratio test

h. $\sum_{n=1}^{\infty} \left(\frac{8n}{3n-2}\right)^n$ $\lim_{n\to\infty} \sqrt[n]{\left(\frac{8n}{3n-2}\right)^n} = \lim_{n\to\infty} \frac{8n}{3n-2} = \frac{8}{3} > 1$

diverges by the root test

8. Find the Taylor Polynomial for the function at the indicated value of c. Use the tables provided. (15 points)

$$f(x) = \cosh x\,, n = 6, c = 0$$

n	n!	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	$\cosh(x)$	1	1	1
1	1	$\sinh(x)$	0	x	0
2	2	$\cosh(x)$	1	x^2	$x^2/2$
3	6	$\sinh(x)$	0	x^3	0
4	24	$\cosh(x)$	1	x^4	$x^4/24$
5	120	$\sinh(x)$	0	x^5	0
6	720	$\cosh(x)$	1	x^6	$x^6/720$

$P_n(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720}$

9. Use the Taylor series expansion of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to find the Taylor series for the function $u(x) = \frac{e^x}{1+x}$. (10 points)

$$1 + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + \tfrac{3}{8}x^4 + \dots$$

$$1+x \overline{\big)\, 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots}$$

$$-1 \mp x$$

$$\frac{x^2}{2} + \frac{x^3}{6}$$

$$-\frac{1}{2}x^2 \mp \frac{x^3}{2}$$

$$-\tfrac{1}{3}x^3 + \frac{x^4}{24}$$

$$\mp \tfrac{1}{3}x \mp x^4/3$$

10. Find the power series for the functions below. Write your answers with the sum starting at n=0. (12 points each)

a. $f(x) = \ln(x-2)$

$$f'(x) = \frac{1}{x-2} \rightarrow \frac{-1/2}{1-\frac{1}{2}x} \qquad a = -1/2,\ r = \tfrac{1}{2}x$$

$$\sum_{n=0}^{\infty} \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}x\right)^n = \sum_{n=0}^{\infty} \left(-\frac{1}{2^{n+1}}\right) x^n \rightarrow$$

$$\sum_{n=0}^{\infty} \frac{-1}{2^{n+1} \cdot (n+1)} x^{n+1}$$

b. $g(x) = \frac{6x}{(1+4x)^2}$

$$\sum ar^n = a(1-r)^{-1}$$

$$\sum_{n=1} anr^{n-1} = a(1-r)^{-2}$$

$$\sum_{n=0}^{\infty} a(n+1)r^n = \frac{a}{(1-r)^2}$$

$$\sum_{n=0}^{\infty} 6x(n+1)(-4x)^n = \sum_{n=0}^{\infty} (n+1)6(-4)^n x^{n+1}$$