1/30/2024

Center of Mass
Exponential and Log integrals
Growth and Decay
Hyperbolic Trig Functions

Center of mass, has some similarities with mean of probability distributions from last time.

To find the center of mass: 1) the total mass, and we also need 2) the moments of mass in each direction (one for x, and one for y).

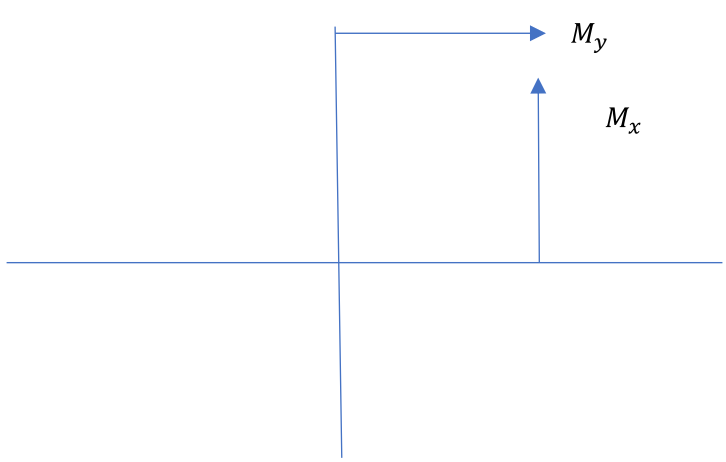
In total, we'll have three integrals to complete.

We assume for these problems that we are dealing with a thin sheet spread over a region bounded by functions, and the mass density is constant (ρ).

The total mass:

$$M = \rho \int_a^b [f(x) - g(x)]dx$$

Moments of mass are subscripted by the axis they run perpendicular to. M_x is the moment of mass from the x-axis, but that means it's actually running in the y-direction.



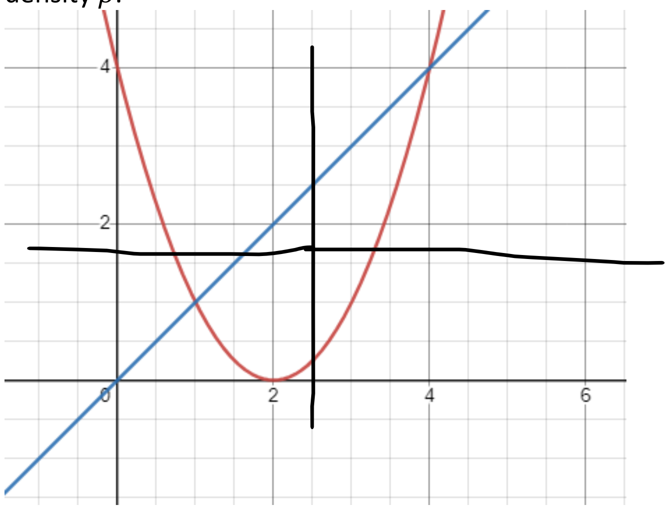
The M_y is the moment of mass from the y-axis, and therefore, running perpendicular to it in the x-direction. We'll use this moment of mass to find the x-component of the center of mass.

$$center\ of\ mass = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

$$M_y = \rho \int_a^b x[f(x) - g(x)]dx$$

$$M_x = \frac{\rho}{2}\int_a^b [f(x)]^2 - [g(x)]^2\,dx$$

Find the center of mass of the region bounded by $y = x^2 - 4x + 4 = (x-2)^2, y = x$ with constant density ρ.



$$M = \rho\int_1^4 x - (x^2 - 4x + 4)dx$$

$$x = x^2 - 4x + 4$$
$$x^2 - 5x + 4 = 0$$
$$(x-1)(x-4) = 0$$
$$x = 1, x = 4$$

$$M = \rho\int_1^4 -x^2 + 5x - 4dx = \rho\left[-\frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x\right]_1^4 = \rho\left[\frac{9}{2}\right] = \frac{9\rho}{2}$$

$$M_y = \rho\int_1^4 x(-x^2 + 5x - 4)dx = \rho\int_1^4 -x^3 + 5x^2 - 4xdx = \rho\left[-\frac{1}{4}x^4 + \frac{5}{3}x^3 - 2x^2\right]_1^4 =$$

$$\rho\left[\frac{45}{4}\right] = \frac{45\rho}{4}$$

$$M_x = \frac{\rho}{2}\int_1^4 x^2 - ((x-2)^2)^2dx = \frac{\rho}{2}\left[\frac{1}{3}x^3 - \frac{1}{5}(x-2)^5\right]_1^4 = \frac{\rho}{2}\left[\frac{72}{5}\right] = \frac{36\rho}{5}$$

Center of mass

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{45\rho}{4}}{\frac{9\rho}{2}} = \frac{45\rho}{4} \times \frac{2}{9\rho} = \frac{5}{2}$$

$$\bar{y} = \frac{M_x}{M} = \frac{36\rho}{5} \times \frac{2}{9\rho} = \frac{8}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{5}{2}, \frac{8}{5}\right) = (2.5, 1.6)$$

Exponential function that are not base e

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

Review of log and exponential rules:

$$e^x e^y = e^{x+y}$$
$$(e^x)^y = e^{xy}$$
$$\frac{e^x}{e^y} = e^{x-y}$$

$$a^x = \left(e^{\ln a}\right)^x = e^{(\ln a)x}$$

$$\log MN = \log M + \log N$$
$$\log\left(\frac{M}{N}\right) = \log M - \log N$$

$$\log M^r = r \log M$$

$$\log_a b = \frac{\ln b}{\ln a}$$

Exponential Growth and Decay:
What is the population after time t? When is the population equal to N? These are just algebra questions.
What is the rate of growth after t years? – Derivative.

Continuous compounding interest:

$$A = Pe^{rt}$$

Newton's Law of Cooling

$$T(t) = (T_0 - T_a)e^{-kt} + T_a$$

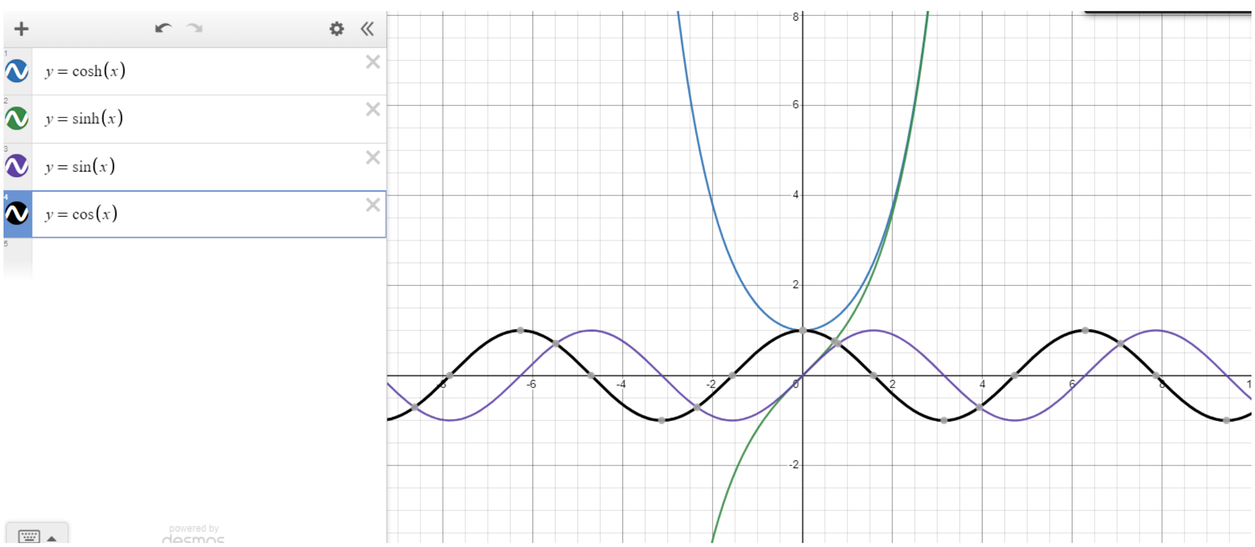
Hyperbolic Trigonometric Functions

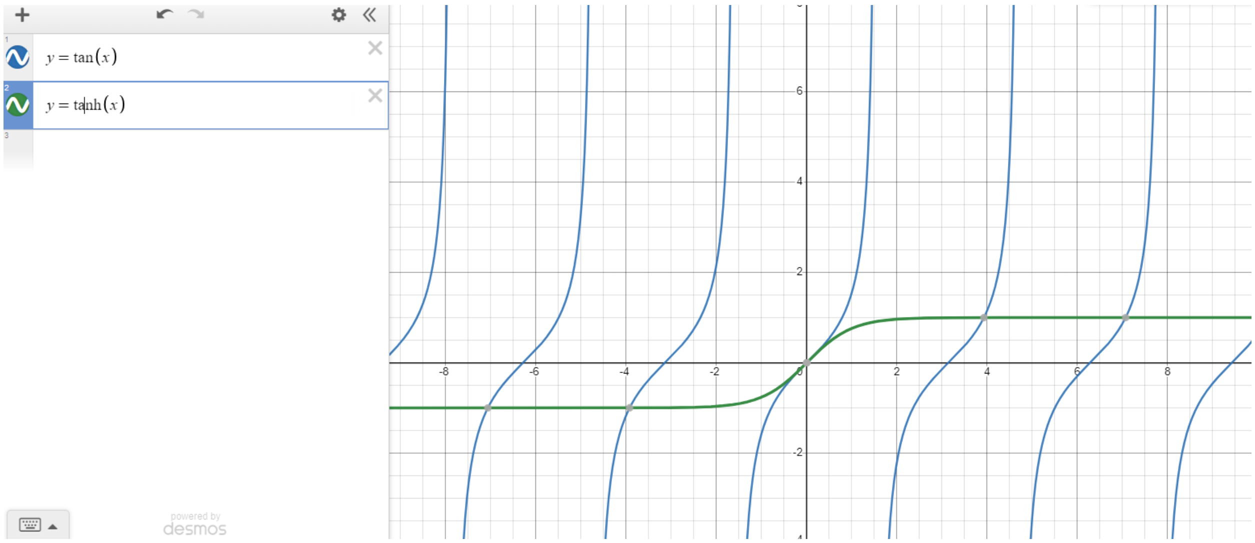
Regular Trig	Hyperbolic Trig
$\sin(x) = \frac{\left(e^{ix} - e^{-ix}\right)}{2i}$	$\sinh(x) = \frac{e^x - e^{-x}}{2}$
$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$	$\cosh(x) = \frac{e^x + e^{-x}}{2}$
$\tan(x) = \frac{\sin(x)}{\cos(x)}$	$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\cot(x) = \frac{1}{\tan(x)}$	$\coth(x) = \frac{1}{\tanh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
$\sec(x) = \frac{1}{\cos(x)}$	$\mathrm{sech}(x) = \frac{1}{\cosh(x)}$
$\csc(x) = \frac{1}{\sin(x)}$	$\mathrm{csch}(x) = \frac{1}{\sinh(x)}$
$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$

$$e^{ix} = \cos(x) + i \sin(x)$$

Regular	Hyperbolic
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\sinh x) = \cosh x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\cosh x) = \sinh x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\tanh x) = \mathrm{sech}^2 x$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\frac{d}{dx}(\coth x) = -\mathrm{csch}^2 x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\mathrm{sech}\, x) = -\mathrm{sech}\, x \tanh x$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\frac{d}{dx}(\mathrm{csch}\, x) = -\mathrm{csch}\, x \coth x$





May be asked to verify an identity or prove a derivative relationship for these hyperbolic trig functions.

If the problem involves sine or cosine (hyperbolic), go back to the exponential identities to do the proof or verification.
If the problem involves (hyperbolic) tangent, cotangent, secant or cosecant, go back to the sine and cosine versions and prove from there (don't use the exponential version)

Prove that the derivative of hyperbolic sine is hyperbolic cosine.

$$\frac{d}{dx}[\sinh x] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{1}{2}\frac{d}{dx}(e^x - e^{-x}) = \frac{1}{2}[e^x + e^{-x}) = \frac{e^x - e^{-x}}{2} = \cosh x$$

Prove that the derivative of hyperbolic tangent is hyperbolic secant squared.

$$\frac{d}{dx}[\tanh x] = \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

Next: Advanced integration techniques:
1) Integration by parts