2/8/2024

Trig Integrals (continued)
Trig Substitution

Sine and cosine:

1) If the power of sine is odd, pull out one sine to be the du, and convert the remaining sines to cosines, u is the cosine.
2) If the power cosine is odd, pull out one cosine to be the du, and convert the remaining cosines to sines, u is the sine.
3) If both powers are even, then use the power-reducing identity until all the terms are linear.

Tangent and secant (or combination of cotangent and cosecant):

1) If you have even numbers of secants, then pull out two secants and convert the remaining secants to tangents, let u be tangent.
2) If the tangents are odd, then pull out one secant and one tangent to be du, and then convert the remaining even tangents to secants, u is secant.
3) If the tangents are even and/or the secants are odd, use integration by parts.

Everything else: combinations of functions that don't share a Pythagorean identity or three or more functions, use identities to convert to sine and cosine and then go from there.

Example.

$$\int \sec^4 x \tan x \, dx$$

$$\int \sec^2 x \tan x \,(\sec^2 x) dx = \int (1 + \tan^2 x) \tan x \,(\sec^2 x) dx$$

$$u = \tan x \,, du = \sec^2 x \, dx$$

$$\int (1 + u^2) u du = \int u + u^3 du = \frac{1}{2}u^2 + \frac{1}{4}u^4 + C = \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$$

Alternatively:

$$\int \sec^3 x \,(\sec x \tan x) dx$$

$$u = \sec x \,, du = \sec x \tan x \, dx$$

$$\int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}\sec^4 x + C$$

Example.

$$\int \sec^3 x \, dx$$

$$u = \sec x \,, dv = \sec^2 x \, dx$$

$$du = \sec x \tan x\, dx, v = \tan x$$

$$\sec x \tan x - \int \sec x \tan^2 x\, dx = \sec x \tan x - \int \sec x\, (\sec^2 x - 1) dx =$$

$$\sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\, dx$$

$$\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\, dx$$

Add the secant-cubed integral to both sides

$$2 \int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$$

Combinations of other functions:

Example.

$$\int \cot x \sec x\, dx = \int \frac{\cos x}{\sin x} \left(\frac{1}{\cos x} \right) dx = \int \frac{1}{\sin x} dx = \int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

Double angle formulas:

$$2 \sin \theta\, cos\theta = \sin 2\theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$$

Trigonometric Substitution

We are trying to integrate function with radicals and squares under the radicals.

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{x^2 + 1}$$

$$\frac{d}{dx}[\text{arcsec}\, x] = \frac{1}{x\sqrt{x^2 - 1}}$$

If your radical is of the form $\sqrt{a^2 - x^2}$, then $a \sin \theta = x$
If your radical is of the form $\sqrt{a^2 + x^2}$, then use $a \tan \theta = x$
If your radical is of the form $\sqrt{x^2 - a^2}$, then use $a \sec \theta = x$

Example. Integrate: $\int \sqrt{1+4x^2}dx$

$$4x^2 = (2x)^2 \to u = 2x, du = 2dx$$

$$\int \sqrt{1+4x^2}dx = \frac{1}{2}\int \sqrt{1+u^2}\,du$$

$$2x = u = \tan\theta\,, du = \sec^2\theta\, d\theta$$

$$\sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \sec x$$

$$\frac{1}{2}\int \sqrt{1+u^2}\,du = \frac{1}{2}\int \sec\theta \sec^2\theta\, d\theta = \frac{1}{2}\int \sec^3\theta\, d\theta$$

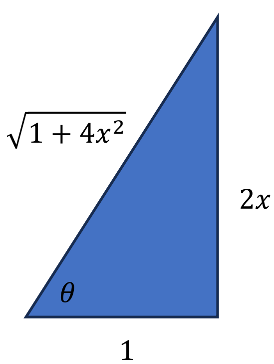
$$\frac{1}{2}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C\right] =$$

$$\frac{2x}{4}\sqrt{1+4x^2} + \frac{1}{4}\ln\left|\sqrt{1+4x^2} + (2x)\right| + C$$

Your final expressions should be in terms of the original variable, and ALL TRIG FUNCTIONS MUST BE SIMPLIFIED…

$\sin(\sin^{-1} x)$ is not allowed in final answers!

$$\frac{2x}{1} = \tan\theta$$



Example. $\int x\sqrt{9-x^2}dx$

$$x = 3\sin\theta\,, dx = 3\cos\theta\, d\theta$$
$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

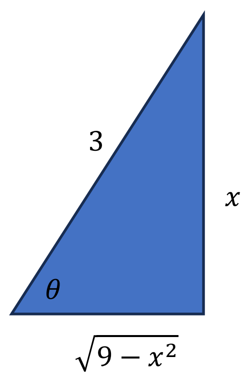
$$\frac{\sqrt{9-x^2}}{3} = \cos\theta$$

$$\int 3\sin\theta \;\; 3\cos\theta \, 3\cos\theta \, d\theta = \int 27\sin\theta\cos^2\theta \, d\theta =$$

$$u = \cos\theta \, , du = \sin\theta \, d\theta$$

$$\int 27u^2 du = \frac{27}{3}\cos^3\theta + C = 9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + C = \frac{9}{27}(9-x^2)^{\frac{3}{2}} + C = \frac{1}{3}\,(9-x^2)^{\frac{3}{2}} + C$$

$$x = 3\sin\theta \rightarrow \sin\theta = \frac{x}{3}$$



Example. $\int \frac{\sqrt{x^2-4}}{x^2} dx$

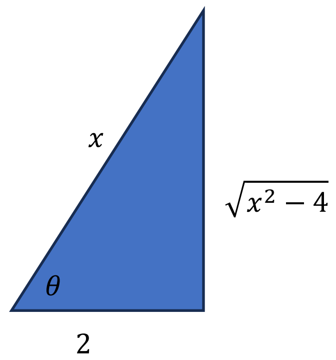
$$x = 2\sec\theta \, , dx = 2\sec\theta\tan\theta \, d\theta$$

$$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = \sqrt{4(\sec^2\theta - 1)} = \sqrt{4\tan^2\theta} = 2\tan\theta$$

$$\int \frac{2\tan\theta \;\; 2\sec\theta\tan\theta \, d\theta}{4\sec^2\theta} = \int \frac{\sec\theta\tan^2\theta}{\sec^2\theta} d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta} d\theta = \int \frac{\sec^2\theta}{\sec\theta} - \frac{1}{\sec\theta} d\theta =$$

$$\int \sec\theta - \cos\theta \, d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

$$\sec\theta = \frac{x}{2}$$



$$\ln|\sec\theta+\tan\theta|-\sin\theta+C=\ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right|-\frac{\sqrt{x^2-4}}{x}+C$$

Can also be use with 3/2 powers.
You may need to complete the square.

Example. $\int \frac{1}{\sqrt{-x^2+10x}}dx$

$$\int \frac{1}{\sqrt{-(x^2-10x)}}dx=\int \frac{1}{\sqrt{-(x^2-10x+25)+25}}dx=\int \frac{1}{\sqrt{25-(x-5)^2}}dx$$

$$x-5=5\sin\theta$$

Example.

$$\int \frac{1}{x^2+2x+1}dx=\int \frac{1}{(x+1)^2}dx$$

This is just a power rule.

$$\int \frac{1}{x^2+2x+5}dx=\int \frac{1}{(x+1)^2+4}dx$$

This is just an arctangent integral.