2. **a)** When $(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$ is expanded, all products of a term in the first sum, a term in the second sum, a term in the third sum, a term in the fourth sum, and a term in the fifth sum are added. Terms of the form x^5, x^4y, x^3y^2, x^2y^3, xy^4 and y^5 arise. To obtain a term of the form x^5, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^5 term in the product has a coefficient of 1. (We can think of this coefficient as $\binom{5}{5}$.) To obtain a term of the form x^4y, an x must be chosen in four of the five sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 4-combinations of five objects, namely $\binom{5}{4} = 5$. Similarly, the number of terms of the form x^3y^2 is the number of ways to pick three of the five sums to obtain x's (and consequently take a y from each of the other two factors). This can be done in $\binom{5}{3} = 10$ ways. By the same reasoning there are $\binom{5}{2} = 10$ ways to obtain the x^2y^3 terms, $\binom{5}{1} = 5$ ways to obtain the xy^4 terms, and only one way (which we can think of as $\binom{5}{0}$) to obtain a y^5 term. Consequently, the product is $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

b) This is explained in Example 2. The expansion is $\binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$. Note that it does not matter whether we think of the bottom of the binomial coefficient expression as corresponding to the exponent on x, as we did in part **(a)**, or the exponent on y, as we do here.

9. Using the Binomial Theorem, we see that the term involving x^{101} in the expansion of $((2x) + (-3y))^{200}$ is $\binom{200}{99}(2x)^{101}(-3y)^{99}$. Therefore the coefficient is $\binom{200}{99}2^{101}(-3)^{99} = -2^{101}3^{99}C(200, 99)$.

12. We just add adjacent numbers in this row to obtain the next row (starting and ending with 1, of course):

1 11 55 165 330 462 462 330 165 55 11 1

9. Let $b_1, b_2, \ldots, b_8$ be the number of bagels of the 8 types listed (in the order listed) that are selected. Order does not matter: we are presumably putting the bagels into a bag to take home, and the order in which we put them there is irrelevant.

a) If we want to choose 6 bagels, then we are asking for the number of nonnegative solutions to the equation $b_1 + b_2 + \cdots + b_8 = 6$. Theorem 2 applies, with $n = 8$ and $r = 6$, giving us the answer $C(8+6-1, 6) = C(13, 6) = 1716$.

b) This is the same as part **(a)**, except that $r = 12$ rather than 6. Thus there are $C(8+12-1, 12) = C(19, 12) = C(19, 7) = 50{,}388$ ways to make the selection. (Note that $C(19, 7)$ was easier to compute than $C(19, 12)$, and since they are equal, we chose the latter form.)

c) This is the same as part **(a)**, except that $r = 24$ rather than 6. Thus there are $C(8+24-1, 24) = C(31, 24) = C(31, 7) = 2{,}629{,}575$ ways to make the selection.

d) This one is more complicated. Here we want to solve the equation $b_1 + b_2 + \cdots + b_8 = 12$, subject to the constraint that each $b_i \geq 1$. We reduce this problem to the form in which Theorem 2 is applicable with the following trick. Let $b_i' = b_i - 1$; then b_i' represents the number of bagels of type i, in excess of the required 1, that are selected. If we substitute $b_i = b_i' + 1$ into the original equation, we obtain $(b_1'+1) + (b_2'+1) + \cdots + (b_8'+1) = 12$, which reduces to $b_1' + b_2' + \cdots + b_8' = 4$. In other words, we are asking how many ways are there to choose the 4 extra bagels (in excess of the required 1 of each type) from among the 8 types, repetitions allowed. By Theorem 2 the number of solutions is $C(8+4-1, 4) = C(11, 4) = 330$.

e) This final part is even trickier. First let us ignore the restriction that there can be no more than 2 salty bagels (i.e., that $b_4 \leq 2$). We will take into account, however, the restriction that there must be at least 3 egg bagels (i.e., that $b_3 \geq 3$). Thus we want to count the number of solutions to the equation $b_1 + b_2 + \cdots + b_8 = 12$, subject to the condition that $b_i \geq 0$ for all i and $b_3 \geq 3$. As in part **(d)**, we use the trick of choosing the 3 egg bagels at the outset, leaving only 9 bagels free to be chosen; equivalently, we set $b_3' = b_3 - 3$, to represent the extra egg bagels, above the required 3, that are chosen. Now Theorem 2 applies to the number of solutions of $b_1 + b_2 + b_3' + b_4 + \cdots + b_8 = 9$, so there are $C(8+9-1, 9) = C(16, 9) = C(16, 7) = 11{,}440$ ways to make this selection.

Next we need to worry about the restriction that $b_4 \le 2$. We will impose this restriction by subtracting from our answer so far the number of ways to violate this restriction (while still obeying the restriction that $b_3 \ge 3$). The difference will be the desired answer. To violate the restriction means to have $b_4 \ge 3$. Thus we want to count the number of solutions to $b_1 + b_2 + \cdots + b_8 = 12$, with $b_3 \ge 3$ and $b_4 \ge 3$. Using the same technique as we have just used, this is equal to the number of nonnegative solutions to the equation $b_1 + b_2 + b_3' + b_4' + b_5 + \cdots + b_8 = 6$ (the 6 on the right being $12 - 3 - 3$). By Theorem 2 there are $C(8 + 6 - 1, 6) = C(13, 6) = 1716$ ways to make this selection. Therefore our final answer is $11440 - 1716 = 9724$.

13. Assuming that the warehouses are distinguishable, let w_i be the number of books stored in warehouse i. Then we are asked for the number of solutions to the equation $w_1 + w_2 + w_3 = 3000$. By Theorem 2 there are $C(3 + 3000 - 1, 3000) = C(3002, 3000) = C(3002, 2) = 4{,}504{,}501$ of them.

15. **a)** Let $x_1 = x_1' + 1$; thus x_1' is the value that x_1 has in excess of its required 1. Then the problem asks for the number of nonnegative solutions to $x_1' + x_2 + x_3 + x_4 + x_5 = 20$. By Theorem 2 there are $C(5 + 20 - 1, 20) = C(24, 20) = C(24, 4) = 10{,}626$ of them.

b) Substitute $x_i = x_i' + 2$ into the equation for each i; thus x_i' is the value that x_i has in excess of its required 2. Then the problem asks for the number of nonnegative solutions to $x_1' + x_2' + x_3' + x_4' + x_5' = 11$. By Theorem 2 there are $C(5 + 11 - 1, 11) = C(15, 11) = C(15, 4) = 1365$ of them.

c) There are $C(5 + 21 - 1, 21) = C(25, 21) = C(25, 4) = 12650$ solutions with no restriction on x_1. The restriction on x_1 will be violated if $x_1 \ge 11$. Following the procedure in part **(a)**, we find that there are $C(5 + 10 - 1, 10) = C(14, 10) = C(14, 4) = 1001$ solutions in which the restriction is violated. Therefore there are $12650 - 1001 = 11{,}649$ solutions of the equation with its restriction.

d) First let us impose the restrictions that $x_3 \ge 15$ and $x_2 \ge 1$. Then the problem is equivalent to counting the number of solutions to $x_1 + x_2' + x_3' + x_4 + x_5 = 5$, subject to the constraints that $x_1 \le 3$ and $x_2' \le 2$ (the latter coming from the original restriction that $x_2 < 4$). Note that these two restrictions cannot be violated simultaneously. Thus if we count the number of solutions to $x_1 + x_2' + x_3' + x_4 + x_5 = 5$, subtract the number of its solutions in which $x_1 \ge 4$, and subtract the numbers of its solutions in which $x_2' \ge 3$, then we will have the answer. By Theorem 2 there are $C(5 + 5 - 1, 5) = C(9, 5) = 126$ solutions of the unrestricted equation. Applying the first restriction reduces the equation to $x_1' + x_2' + x_3' + x_4 + x_5 = 1$, which has $C(5 + 1 - 1, 1) = C(5, 1) = 5$ solutions. Applying the second restriction reduces the equation to $x_1 + x_2'' + x_3' + x_4 + x_5 = 2$, which has $C(5 + 2 - 1, 2) = C(6, 2) = 15$ solutions. Therefore the answer is $126 - 5 - 15 = 106$.

23. There are several ways to count this. We can first choose the two objects to go into box #1 ($C(12, 2)$ ways), then choose the two objects to go into box #2 ($C(10, 2)$ ways, since only 10 objects remain), then choose the two objects to go into box #3 ($C(8, 2)$ ways), and so on. So the answer is $C(12, 2) \cdot C(10, 2) \cdot C(8, 2) \cdot C(6, 2) \cdot C(4, 2) \cdot C(2, 2) = (12 \cdot 11/2)(10 \cdot 9/2)(8 \cdot 7/2)(6 \cdot 5/2)(4 \cdot 3/2)(2 \cdot 1/2) = 12!/2^6 = 7{,}484{,}400$. Alternatively, just line up the 12 objects in a row (12! ways to do that), and put the first two into box #1, the next two into box #2, and so on. This overcounts by a factor of 2^6, since there are that many ways to swap objects in the permutation without affecting the result (swap the first and second objects or not, and swap the third and fourth objects or not, and so on). So this results in the same answer. Here is a third way to get this answer. First think of pairing the objects. Think of the objects as ordered (a first, a second, and so on). There are 11 ways to choose a mate for the first object, then 9 ways to choose a mate for the first unused object, then 7 ways to choose a mate for the first still unused object, and so on. This gives $11 \cdot 9 \cdot 7 \cdot 5 \cdot 3$ ways to do the pairing. Then there are 6! ways to choose the boxes for the pairs. So the answer is the product of these two quantities, which is again 7,484,400.

31. This is a direct application of Theorem 3, with $n = 11$, $n_1 = 5$, $n_2 = 2$, $n_3 = n_4 = 1$, and $n_5 = 2$ (where n_1 represents the number of A's, etc.). Thus the answer is $11!/(5!2!1!1!2!) = 83{,}160$.

41. This is like Example 9. If we approach it as is done there, we see that the answer is

$$C(52,7)C(45,7)C(38,7)C(31,7)C(24,7) = \frac{52!}{7!45!} \cdot \frac{45!}{7!38!} \cdot \frac{38!}{7!31!} \cdot \frac{31!}{7!24!} \cdot \frac{24!}{7!17!} = \frac{52!}{7!7!7!7!7!17!} \approx 7.0 \times 10^{34}.$$

Applying Theorem 4 will yield the same answer; in this approach we think of the five players and the undealt cards as the six distinguishable boxes.

55. Since each box has to contain at least one object, we might as well put one object into each box to begin with. This leaves us with just two more objects, and there are only two choices: we can put them both into the same box (so that the partition we end up with is $6 = 3 + 1 + 1 + 1$), or we can put them into different boxes (so that the partition we end up with is $6 = 2 + 2 + 1 + 1$). So the answer is 2.

. 156423, 165432, 231456, 231465, 234561, 314562, 432561, 435612, 541236, 543216, 654312, 654321

7. We begin with the first 3-combination, namely $\{1,2,3\}$. Let us trace through Algorithm 3 to find the next. Note that $n = 5$ and $r = 3$; also $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$. We set i equal to 3 and then decrease i until $a_i \neq 5 - 3 + i$. This inequality is already satisfied for $i = 3$, since $a_3 \neq 5$. At this point we increment a_i by 1 (so that now $a_3 = 4$), and fill the remaining spaces with consecutive integers following a_i (in this case there are no more remaining spaces). Thus our second 3-combination is $\{1,2,4\}$. The next call to Algorithm 3 works the same way, producing the third 3-combination, namely $\{1,2,5\}$. To get the fourth 3-combination, we again call Algorithm 3. This time the i that we end up with is 2, since $5 = a_3 = 5 - 3 + 3$. Therefore the second element in the list is incremented, namely goes from a 2 to a 3, and the third element is the next larger element after 3, namely 4. Thus this 3-combination is $\{1,3,4\}$. Another call to the algorithm gives us $\{1,3,5\}$, and another call gives us $\{1,4,5\}$. Now when we call the algorithm, we find $i = 1$ at the end of the **while** loop, since in this case the last two elements are the two largest elements in the set. Thus a_1 is increased to 2, and the remainder of the list is filled with the next two consecutive integers, giving us $\{2,3,4\}$. Continuing in this manner, we get the rest of the 3-combinations: $\{2,3,5\}$, $\{2,4,5\}$, $\{3,4,5\}$.

11. One way to do this problem (and to have done Exercise 10) is to generate the r-combinations using Algorithm 3, and then to find all the permutations of each, using Algorithm 1 (except that now the elements to be permuted are not the integers from 1 to r, but are instead the r elements of the r-combination currently being used). Thus we start with the first 3-combination, $\{1,2,3\}$, and we list all 6 of its permutations: 123, 132, 213, 231, 312, 321. Next we find the next 3-combination, namely $\{1,2,4\}$, and list all of its permutations: 124, 142, 214, 241, 412, 421. We continue in this manner to generate the remaining 48 3-permutations of $\{1,2,3,4,5\}$: 125, 152, 215, 251, 512, 521; 134, 143, 314, 341, 413, 431; 135, 153, 315, 351, 513, 531; 145, 154, 415, 451, 514, 541; 234, 243, 324, 342, 423, 432; 235, 253, 325, 352, 523, 532; 245, 254, 425, 452, 524, 542; 345, 354, 435, 453, 534, 543. There are of course $P(5,3) = 5 \cdot 4 \cdot 3 = 60$ items in our list.