Absolute/Relative Extrema Key Math 254 (Summer '12)

1. $f(x,y) = xy$

$f_x = y = 0 \qquad (0,0)$

$f_y = x = 0$

$f_{xx} = 0 \qquad D = 0 \cdot 0 - (1)^2 = -1$

$f_{yy} = 0 \qquad$ saddle pt.

$f_{xy} = 1$

2. $f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$

$f_x = 4x + 2y + 2 = 0 \qquad 4x - 2x + 2 = 0 \quad 2x = -2 \quad x = -1$

$f_y = 2x + 2y = 0 \quad x = -y \qquad y = 1$

$f_{xx} = 4 \qquad (-1, 1)$

$f_{yy} = 2 \qquad D = 4 \cdot 2 - 2^2 = 8 - 4 = 4$

$f_{xy} = 2 \qquad f_{xx} > 0 \quad$ min

3. $f(x,y) = (x^2 + y^2)^{1/2}$

$f_x = \dfrac{x}{\sqrt{x^2+y^2}} \qquad x = 0$

$f_y = \dfrac{y}{\sqrt{x^2+y^2}} \qquad y = 0$

$$f_{xx} = \frac{1\sqrt{x^2+y^2} - \frac{1}{2}x \cdot 2x \frac{1}{\sqrt{x^2+y^2}}}{(x^2+y^2)} \cdot \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}$$

$f_{yy} = \dfrac{x^2}{\left(\sqrt{x^2+y^2}\right)^3}$

D not defined

$f_{xy} = \dfrac{-xy}{\left(\sqrt{x^2+y^2}\right)^3} \qquad$ inconclusive

geometrically, it is a minimum

4. $f(x,y) = (x^2+4y^2)e^{1-x^2-y^2}$

$f_x = (2x)e^{1-x^2-y^2} + (x^2+4y^2)e^{1-x^2-y^2}(-2x)$

$e^{1-x^2-y^2}\left[2x\left[1-x^2-4y^2\right]\right] = 0 \qquad x=0 \qquad x^2+4y^2=1$

$f_y = \cancel{8}y e^{1-x^2-y^2} + (x^2+4y^2)e^{1-x^2-y^2}(-2y)$

$2y\,e^{1-x^2-y^2}\left[4-x^2-4y^2\right] = 0 \qquad y=0 \qquad x^2+4y^2=4$

not both true

$(x=0, y=0)$ $\qquad x=0 \quad \cancel{x^2}+4y^2=4 \qquad\qquad y=0,\ x^2+\cancel{4y^2}=1$

$(0,0) \qquad\qquad y=\pm 1 \qquad\qquad x=\pm 1$

$(0,1)\ (0,-1) \qquad\qquad (1,0)\ (-1,0)$

$f_{xx} = e^{1-x^2-y^2}\left[2x\left[1-x^2-4y^2\right]\right](-2x) + e^{1-x^2-y^2}[2]\left[1-x^2-4y^2\right] +$
$e^{1-x^2-y^2}\left[2x\left[-2x\right]\right]$

$f_{yy} = 2e^{1-x^2-y^2}\left[4-x^2-4y^2\right] + 2y\,e^{1-x^2-y^2}(-2y)\left[4-x^2-4y^2\right] +$
$2y\,e^{1-x^2-y^2}\left[\cancel{4}-8y\right]$

$f_{xy} = e^{1-x^2-y^2}(-2y)\left[2x\left[1-x^2-4y^2\right]\right] + e^{1-x^2-y^2}\left[2x(-8y)\right]$

$D(0,0) = \left[\cancel{(e)(0)} + (e)(2)(1)\right]\left[2(e)(4) + \cancel{0+0}\right] - \left[(e)(0+0)\right]^2$

$2e \cdot 8e = 16e^2 > 0 \qquad$ min

$D(0,1) = \left[\cancel{1(0)(-3)} + (1)(2)(-3) + \cancel{(1)(0)}\right]\left[\cancel{2(1)(0)} + \cancel{2(1)(4)(1)(0)} + 2(1)(1)(-8)\right]$
$-\left[\cancel{(1)(-2)(0)(-3)} + \cancel{(1)(0)(-8)}\right]^2 = -6 \cdot (-16) > 0 \qquad$ max

$D(0,-1) = \left[0 + (1)(2)(-3) + 0\right]\left[\cancel{2(1)}(0) + 0 + 2(1)(-8)\right] - \left[0+0\right]^2 = -6 * -16 > 0$ max

$D(1,0) = \left[\cancel{1(2)(0)(1)} + \cancel{(1)(2)}(0) + (1)(2)(-2)\right]\left[2(1)(3) + 0 + 0\right] - \left[0+0\right]^2 = -4(6) < 0$

Saddle

$D(-1,0) = \left[\cancel{1(-2)}(0) + 0 + (1)(2)(-2)\right]\left[2(1)(3) + 0 + 0\right] - \left[0+0\right]^2 = -4(6) < 0$

Saddle

5. $f(x,y) = x^3 + y^3$

$f_x = 3x^2 = 0 \quad x = 0$

$f_y = 3y^2 = 0 \quad y = 0 \quad (0,0)$

$f_{xx} = 6x$

$f_{yy} = 6y \quad D = 0 \cdot 0 - 0^2 = 0$ inconclusive

$f_{xy} = 0$ graphically it's none

6. $f(x,y) = x^{2/3} + y^{2/3}$

$f_x = \frac{2}{3}x^{-1/3}$ undef. at 0 $(0,0)$

$f_y = \frac{2}{3}y^{-1/3}$ undef at 0

$f_{xy} = -\frac{2}{9}x^{-4/3}$

$f_{yy} = -\frac{2}{9}y^{-4/3} \quad D =$ undefined

$f_{xy} = 0$ graphically min

7. $f(x,y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

$f_x = 3x^2 - 12x + 12 = 0 \quad x^2 - 4x + 4 = 0 \quad (x-2)^2 = 0 \quad x = 2$

$f_y = 3y^2 + 18y + 27 = 0 \quad y^2 + 6y + 9 = 0 \quad (y+3)^2 = 0 \quad y = -3$

$f_{xx} = 6x - 12$ at $(2,-3) = 0$

$f_{yy} = 6y + 18$ at $(2,-3) = 0$

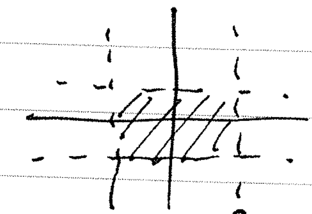
$f_{xy} = 0 \quad D = 0 \cdot 0 - 0 = 0$ inconclusive

graphically none

8. $f(x,y,z) = x^2 + (y-3)^2 + (z+1)^2$

graphically this is a 4D paraboloid (hyperparaboloid) centered at $(0,3,-1)$ it opens "up" so this is a minimum

9. $f(x,y) = x^2 + xy \quad R: \{(x,y) \mid -2 \le x \le 2,\ -1 \le y \le 1\}$



$f_x = 2x + y \quad y = 0$

$f_y = x = 0 \quad (0,0)$

$f(-2,y) = 4 - 2y \qquad f(x,-1) = x^2 - x$

$f'(-2,y) = -2 \ne 0 \qquad f'(x,-1) = 2x - 1 = 0 \quad x = \frac{1}{2} \qquad f(0,0) = 0$

$f(2,y) = 4 + 2y \qquad f(x,1) = x^2 + x \qquad f(\frac{1}{2},-1) = -\frac{1}{4}$

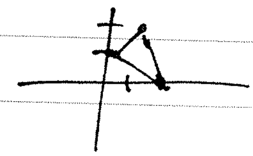
$f'(2,y) = 2 \ne 0 \qquad f'(x,1) = 2x + 1 = 0 \quad x = -\frac{1}{2} \qquad f(-\frac{1}{2},1) = -\frac{1}{4}$

corners $f(-2,-1) = 6 \quad f(-2,1) = 2 \quad f(2,-1) = 2 \quad f(2,1) = 6$

Abs min at $f(0,0) = 0$ abs max at $f(-2,-1) = f(2,1) = 6$

10. $f(x,y) = 12 - 3x - 2y$ R bounded by triangle w/ vertices $(2,0), (0,1), (1,2)$

$f_x = -3 \ne 0$ no critical points



$f_y = -2 \ne 0$

boundaries: $m = \frac{1-0}{0-2} = -\frac{1}{2} \quad y = -\frac{1}{2}x + 1$

$m = \frac{2-1}{1-0} = 1 \quad y = x + 1$

$m = \frac{2-0}{1-2} = -2 \quad y = -2x + 4$

$f(x, -\frac{1}{2}x + 1) = 12 - 3x - 2(-\frac{1}{2}x + 1) = 12 - 3x + x - 2 = 10 - 2x \quad f' = -2 \ne 0$

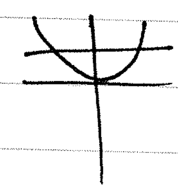
$f(x, x+1) = 12 - 3x - 2(x+1) = 12 - 3x - 2x - 2 = 10 - 5x \quad f' = -5 \ne 0$

$f(x, -2x+4) = 12 - 3x - 2(-2x+4) = 12 - 3x + 4x - 8 = 4 + x \quad f' = 1 \ne 0$

$f(2,0) = 6 \quad f(0,1) = 10 \quad f(1,2) = 5$

MAX MIN

11. $f(x,y) = 2x - 2xy + y^2 \quad R: \{(x,y) \mid y \ge x^2,\ y \le 1\}$



$f_x = 2 - 2y = 0 \quad y = 1 \qquad (1,1)$

$f_y = -2x + 2y = 0 \quad y = x \quad x = 1$

$f(x, x^2) = 2x - 2x^3 + x^4$

$f'(x, x^2) = 2 - 6x^2 + 4x^3 = 0 \qquad x = -\frac{1}{2} \quad x = 1 \quad (-\frac{1}{2}, \frac{1}{4})$

$y = \frac{1}{4} \quad y = 1, y = -1$

11. con't'd

$f(x,1) = 2x - 2x + 1 = 1 \qquad f'(x,1) = 0$

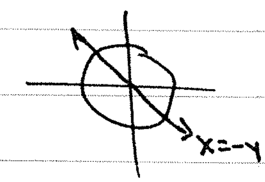
Corner $(-1,+1)$

$f(1,1) = 1$ MAX $\qquad f(-\frac{1}{2}, \frac{1}{4}) = -{}^{11}/_{16}$ MIN $\qquad f(-1,1) = 1$ MAX

12. $f(x,y) = x^2 + 2xy + y^2 \qquad R: \{(x,y) \mid x^2 + y^2 \le 8\}$

$f_x = 2x + 2y = 0 \qquad x = -y$

$f_y = 2x + 2y = 0$



$x^2 + y^2 = 8 \qquad y^2 = 8 - x^2 \qquad y = \pm\sqrt{8-x^2}$

$f(x, \pm\sqrt{8-x^2}) = 8 \pm 2x\sqrt{8-x^2}$

$f'_x(x, \pm\sqrt{8-x^2}) = \pm 2\sqrt{8-x^2} \pm \frac{2x(\frac{1}{2})}{\sqrt{8-x^2}} \cdot -2x = 0$

$\cancel{\sqrt{8-x^2}} \pm 2\sqrt{8-x^2} = \pm \frac{2x^2}{\sqrt{8-x^2}} \cancel{\sqrt{8-x^2}}$

$\cancel{\pm} 2(8-x^2) = \cancel{\pm}\cancel{2}x^2 \qquad 8 - x^2 = x^2 \qquad 8 = 2x^2 \qquad 4 = x^2 \qquad x = \pm 2$

$y = \pm 2$

$f(2,2) = 16$ MAX $\qquad f(2,-2) = 0 \qquad f(-2,2) = 0 \qquad f(-2,-2) = 16$ MAX

$f(x,-y) = 0$ MIN is every point on this line $y = -x$ between $x = -2$ and $x = 2$