Math 2568, Exam #2, Part 1, Summer 2013 Name KEY

Instructions: On this portion of the exam, you may **NOT** use a calculator. Show all work. Answers must be supported by work to receive full credit.

1. Compute the determinant by the cofactor method. (10 points)

$$\begin{vmatrix}1&-1&0&-2\\0&1&2&0\\1&-2&0&1\\7&1&1&-1\end{vmatrix}$$

$$1\begin{vmatrix}1&0&-2\\1&0&1\\7&1&-1\end{vmatrix} - 2\begin{vmatrix}1&-1&-2\\1&-2&1\\7&1&-1\end{vmatrix}$$

$$1\left[-1\begin{vmatrix}1&-2\\1&+1\end{vmatrix}\right] - 2\left[1\begin{vmatrix}-2&1\\1&-1\end{vmatrix}+1\begin{vmatrix}1&1\\7&-1\end{vmatrix}-2\begin{vmatrix}1&-2\\7&1\end{vmatrix}\right]=$$

$$-1(+1+2) - 2[1(2-1)+1(-1-7)-2(1+14)]=$$

$$-1(3) - 2(1-8-30) = -3+74 = \boxed{71}$$

2. Compute the determinant by using row operations. (7 points)

$3 \;\; 0 \;\; -3 \;\; 3$

$$\begin{vmatrix}0&5&-1&7\\2&0&-2&2\\-3&2&1&-3\\0&1&2&4\end{vmatrix}$$

$R_1 \leftrightarrow R_2$ (-1)

$\frac{1}{2}R_1 \to R_1$ (2)

$3R_1+R_3 \to R_3$ (1)

$$\begin{vmatrix}1&0&-1&1\\0&5&-1&7\\0&2&-2&0\\0&1&2&4\end{vmatrix}$$

$R_2 \leftrightarrow R_4$ (-1)

$0 \;\; -2 \;\; -4 \;\; -8$

$0 \;\; -5 \;\; -10 \;\; -20$

$$\begin{vmatrix}1&0&-1&1\\0&1&2&4\\0&2&-2&0\\0&5&-1&7\end{vmatrix}$$

$-2R_2+R_3 \to R_3$ (1)

$-5R_2+R_4 \to R_4$ (1)

$$\begin{vmatrix}1&0&-1&1\\0&1&2&4\\0&0&-6&-8\\0&0&-11&-13\end{vmatrix}$$

$$(-1)(2)(-1)(1)(1)\left(-6\times-13-(-8)\times(-11)\right)=$$

$$2(\cancel{78}-88) = 2(-10) = \boxed{-20}$$

3. Given that A and B are $n \times n$ matrices with det A = -4 and det B = 3, find the following. (3 points each)

a) det (AB) $(-4)(3) = -12$

d) det (B^4) $= 3^4 = 81$

b) det (A^{-1}) $-\frac{1}{4}$

e) det (⅖A) $\left(\frac{2}{5}\right)^n(-4)$

c) det ($-AB^{-1}$)

$(-1)^n(-4)\left(\frac{1}{3}\right) = (-1)^{n+1}\left(\frac{4}{3}\right)$

f) det (A^TBA) $(-4)(3)(-4) = 48$

4. Assume that $A = \begin{bmatrix} 1 & 0 & 1 & -2 & 7 \\ 2 & 1 & -3 & 1 & 2 \\ -2 & 2 & 0 & 3 & 1 \\ 3 & 5 & -1 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & -1 & 4 & 1 \\ 0 & 16 & -2 & 17 & 5 \\ 0 & 0 & 42 & -13 & -33 \\ 0 & 0 & 0 & 17 & -113 \end{bmatrix}$ and $C = \begin{bmatrix} 51 & 0 & 0 & 0 & -176 \\ 0 & 51 & 0 & 0 & 358 \\ 0 & 0 & 51 & 0 & -145 \\ 0 & 0 & 0 & 17 & -113 \end{bmatrix}$ are row equivalent.

a. Find a basis for the column space of *A*. (5 points)

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \\ 4 \end{bmatrix} \right\}$$

b. Find a basis for the null space of A. (7 points)

$x_1 = +176/51\ x_5$

$x_2 = -358/51\ x_5$

$x_3 = 145/51\ x_5$

$x_4 = 113/17\ x_5$

$x_5 = x_5$

$$\text{Nul } A = \left\{ \begin{bmatrix} 176 \\ -358 \\ 145 \\ 339 \\ 51 \end{bmatrix} \right\}$$

5. Determine if each statement is True or False. (2 points each)

a. T F Det(-1A) is always equal to -1Det(A). $(-1)^n \neq 1$ for odd n

b. T F Row operations do not change the determinant of a matrix.
false

c. T F The determinant of a triangular matrix is the sum of the entries on the diagonal.
product

d. T F If $\{\vec{u}, \vec{v}, \vec{w}, \vec{x}, \vec{y}, \vec{z}\}$ is linearly independent, then none of the vectors are in R^5 or any smaller dimension space.

e. T F If A and B are $m \times n$ matrices, then both AB^{T} and $A^{\mathrm{T}}B$ are defined.

f. T F Det(A+B) = Det(A) + Det(B). ex. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ det = 1 $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ det = 1

sum = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ det = 0

$1+1 \neq 0$

g. T F If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, then so is $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ where k<p.

h. T F The pivot columns of a matrix are always linearly independent.

i. T F The set of all even functions is an example of a vector space.

j. T F If A and B are row equivalent, then their column spaces are the same. row operations change column vectors.

k. T F The vector space $\mathbb{P}_n$ and R^{n+1} are isomorphic.

l. T F A linearly dependent set in a subspace H that spans the space is a basis for H.
In-

m. T F The null space of a matrix is a subspace of the domain of the matrix.

n. T F There are only three conditions a vector space must satisfy: it must be closed under addition, closed under multiplication, and must contain the zero vector.

o. T F The kernel of a transformation is the space in the range that the matrix maps onto.
Kernel = Nullspace in domain.

p. T F The third standard basis vector $\vec{e_3}$ in R^6 is $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Math 2568, Exam #2, Part 2, Summer 2013 Name ______________________

Instructions: On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

1. Determine if the following sets are linearly independent or dependent. If the sets are dependent, find a basis for the subspace spanned by the vectors. Is the set a basis for the entire vector space? ($\mathbb{R}^4$ or $\mathbb{R}^3$ or $\mathbf{P}_3$ respectively) (4 points each)

a. $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \cancel{\begin{bmatrix} 3 \\ 3 \\ -6 \\ 7 \end{bmatrix}} \right\}$

to many vectors to be independent $\Rightarrow$ dependent

Basis $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \end{bmatrix} \right\}$ spans $\mathbb{R}^4$ is basis for all of $\mathbb{R}^4$

b. $\left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix} \right\} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

independent is a basis for $\mathbb{R}^3$

c. $\{t-4, 6t+t^2-t^3, t+1, t^3+2\}$

$\begin{bmatrix} -4 & 0 & 1 & 2 \\ 1 & 6 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \sim I_4$ independent

basis

2. Consider the linear transformation $T: p(t) \in P_4 \mapsto P(t) \in P_5$ defined by $T(p(t)) = P(t) = \int_0^t p(x)dx$. Use this information to answer the following questions.

   a. If a typical polynomial in P_4 is given by $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$, write $p(t)$ as a vector $\vec{p}$ in R^5. (2 points)

$$\vec{p} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

   b. Perform the transformation as defined on $p(t)$ to obtain $P(t)$ and write the resulting polynomial as a vector $\vec{P}$ in R^6. (6 points)

$$\int_0^t a_0 + a_1 \cancel{x} + a_2 x^2 + a_3 x^3 + a_4 x^4 \, dx =$$

$$a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \frac{a_3}{4} x^4 + \frac{a_4}{5} x^5 \Big|_0^t =$$

$$a_0 t + \frac{a_1}{2} t^2 + \frac{a_2}{3} t^3 + \frac{a_3}{4} t^4 + \frac{a_4}{5} t^5 = P(t)$$

$$\vec{P} = \begin{bmatrix} 0 \\ a_0 \\ a_1/2 \\ a_2/3 \\ a_3/4 \\ a_4/5 \end{bmatrix}$$

   c. Given that you are mapping from $R^5 \mapsto R^6$, how big will the matrix of this linear transformation be? (2 points)

6x5

   d. Using this information, write a linear transformation matrix A that can map $A\vec{p} = \vec{P}$. (5 points)

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1/5 \end{bmatrix} = A$$

e. Is your matrix one-to-one or onto (or both or neither)? Justify your answer. (4 points)

one-to-one, but not onto

pivots in every column, but not every row.

3. Given the basis for $\mathbb{R}^3$ to be $\mathscr{B}=\left\{\begin{bmatrix}2\\0\\5\end{bmatrix},\begin{bmatrix}-1\\1\\4\end{bmatrix},\begin{bmatrix}3\\-2\\0\end{bmatrix}\right\}$, find the representation of the three standard basis vectors $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ in the new basis. Notate each solution as $[\vec{e_1}]_B = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$ (8 points)

$$P_B = \begin{bmatrix}2 & -1 & 3\\0 & 1 & -2\\5 & 4 & 0\end{bmatrix}$$

$$[\vec{e}_1]_B = P_B^{-1}\vec{e}_1 = \begin{bmatrix}8/11\\-10/11\\-5/11\end{bmatrix}$$

$$[\vec{e}_2]_B = P_B^{-1}\vec{e}_2 = \begin{bmatrix}12/11\\-15/11\\-13/11\end{bmatrix}$$

$$[\vec{e}_3]_B = P_B^{-1}\vec{e}_3 = \begin{bmatrix}-1/11\\4/11\\2/11\end{bmatrix}$$

4. Given the vector $[\vec{x}]_B = \begin{bmatrix} 7 \\ -2 \\ 3 \\ 1 \end{bmatrix}$, find the vector $\vec{x}$ in the standard basis given the basis

$\mathscr{B}=\left\{\begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\}$. (5 points)

$$P_B = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix} \qquad P_B[\vec{x}]_B = \vec{x} = \begin{bmatrix} 14 \\ 10 \\ -5 \\ 16 \end{bmatrix}$$

5. Prove that the following subsets are, or are not, vector spaces. (6 points each)

a. $H = \left\{\begin{bmatrix} r \\ s \\ t \end{bmatrix} : 3r - 2 = 3s + t\right\}$

not a vector space.
fails for no $\vec{0}$

if $s=t=0$
$3r-2=0 \Rightarrow r = 2/3$ $\quad \begin{bmatrix} 2/3 \\ 0 \\ 0 \end{bmatrix}$

if $r=s=0 \Rightarrow t=-2$ $\quad \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

if $r=t=0 \Rightarrow -2=3s$
$s=-2/3$ $\quad \begin{bmatrix} 0 \\ -2/3 \\ 0 \end{bmatrix}$

b. $V = \{set\ of\ polynomials\ defined\ by\ at + bt^2 + ct^5 : a, b, c\ real\}$

isomorphic to vector $\begin{bmatrix} 0 \\ a \\ b \\ 0 \\ 0 \\ c \end{bmatrix}$

1) if $a=b=c=0$ $\quad \vec{0}$ ✓ $\quad p(t)=0$

2) $k\,p(t) = (ka)t + (kb)t^2 + (kc)t^5$, ka, kb, kc real

3) $p(t)+q(t) = (at + bt^2 + ct^5) + (dt + et^2 + ft^5) =$
$(a+d)t + (b+e)t^2 + (c+f)t^5$, $(a+d), (b+e), (c+f)$ real

is a vector space.

6. For each of the following questions, provide a short explanation with theoretical justifications. (4 points each)

a. Explain why det(AB)=det(BA) but AB≠BA.

$\det(AB) = \det A \cdot \det B$. These are real #'s and commute $\therefore$ $\det A \cdot \det B = \det B \cdot \det A = \det(BA)$

but $AB \neq BA$ generally.

eg. $\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \cdot \underset{B}{\begin{bmatrix} 1 & -1 \\ -2 & -5 \end{bmatrix}} = \underset{AB}{\begin{bmatrix} -3 & -11 \\ -5 & -23 \end{bmatrix}}$ but $BA = \underset{BA}{\begin{bmatrix} -2 & -2 \\ -17 & -24 \end{bmatrix}}$

note $\det(AB) = 14$ but $\det(BA) = 14$ also.

b. How are the number of vectors in Nul A related to the number of vectors in Col A? Explain why, in your own words, that this should be true.

number of vectors in Nul A relates to the number of free variables which are found in columns of a matrix that have no pivots. ie. $= n - \#$ of pivots.

but Col A vectors are obtained from columns w/ pivots

So $\underset{\text{(free variables)}}{\#\text{ of vectors in Nul }A} + \underset{\text{(\# of pivots)}}{\#\text{ of vectors in Col }A} = \underset{\text{(\# of columns)}}{n}$