Math 2415, Exam #1, Summer 2014 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Solve the exact equation $(e^x \sin y + y^2)dx + (e^x \cos y + 2xy - 1)dy = 0$. (10 points)

$$\int e^x \sin y + y^2 \, dx = e^x \sin y + xy^2 + f(y)$$

$$\int e^x \cos y + 2xy - 1 \, dy = e^x \sin y + xy^2 - y + g(x)$$

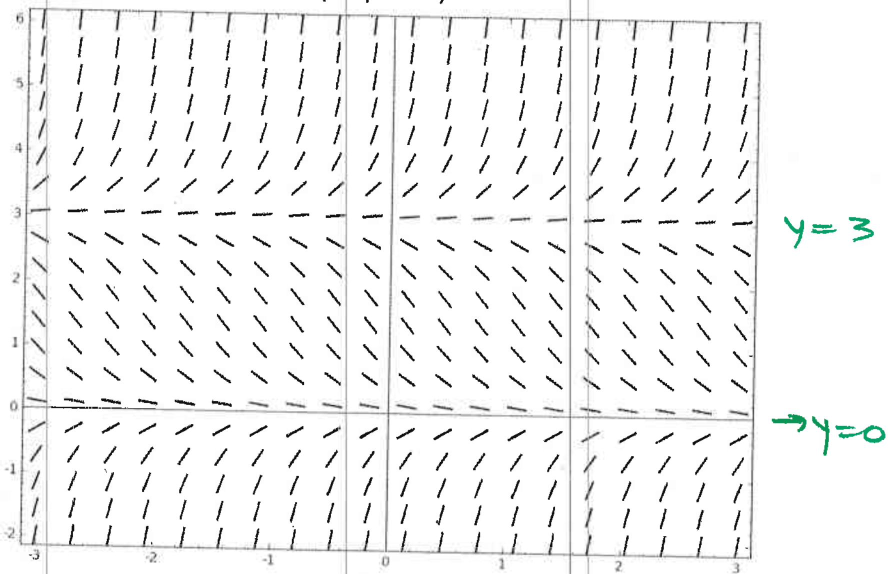
$$\varphi(x,y) = e^x \sin y + xy^2 - y + K = 0$$

2. Define the conditions needed to determine if an equilibrium point in an autonomous population equation is a carrying capacity or a threshold. (7 points)

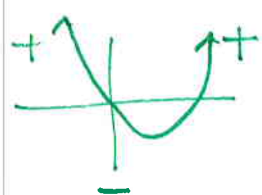
$y > 0$ unstable equilibria are thresholds (collapses below, grows above)

$y > 0$ stable equilibria are carrying capacities (collapses above, grows below)

3. Assuming the equilibrium solutions are integers, use the graph below to sketch the phase portrait of the differential equation that produced the slope field shown here, and write the differential equation that produced it. (10 points)



$$\frac{dy}{dt} = y(y-3)$$



4. For the nonlinear differential equation $y' = \frac{1-2\cos t}{(1+y^3)(3t)}$. Determine the region where the solution is not defined, and then sketch it in the plane. (7 points)

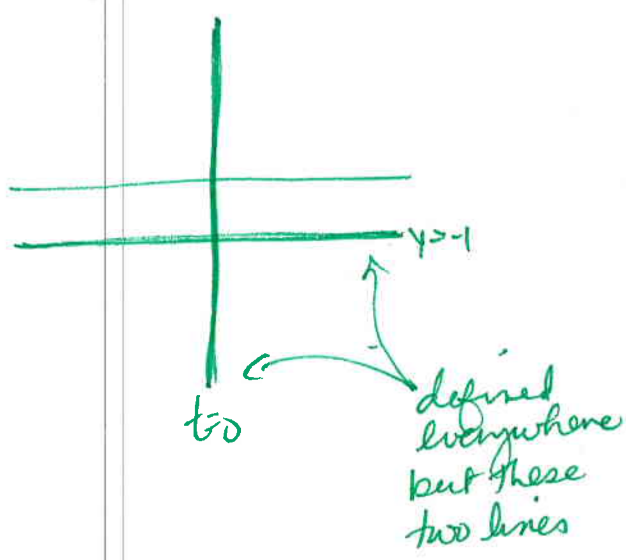
$1+y^3=0 \Rightarrow y=-1$

$3t=0 \Rightarrow t=0$

undefined at this point

$$\frac{df}{dy} = \frac{1-2\cos t}{3t}\left((-1)\,3y^2\,(1+y^3)^{-2}\right)$$

no new points



5. A tank has pure water flowing into it at 20 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 20 L/min. Initially, the tank contains 10 kg of salt in 1000 L of water. Find an equation to model the amount of salt in the tank at any time t. How much salt will there be in the tank after 30 minutes? (14 points)

$A(0) = 10\,kg$

$$\frac{dA}{dt} = Rate_{in} - Rate_{out}$$

$$= \frac{20L}{m}\cdot\frac{0}{L} - \frac{20L}{min}\cdot\frac{A}{1000L} \Rightarrow \frac{dA}{dt} = -\frac{1}{50}A$$

$$\int \frac{dA}{A} = \int -\frac{1}{50}\,dt$$

$$\ln A = -\frac{1}{50}t + C \Rightarrow \quad A = A_0 e^{-1/50 t}$$

$A_0 = 10$

$$A(t) = 10e^{-1/50 t}$$

$$A(30) = 10e^{-\frac{30}{50}} \approx 5.488\ kg$$

6. Solve the differential equation $y' = \frac{x^2+xy+y^2}{x^2}$. (12 points)

$y = vx$

$y' = v'x + v$

$v = \frac{y}{x}$

$$v'x + v = \frac{x^2 + x^2v + v^2x^2}{x^2}$$

$$v'x + \cancel{v} = 1 + \cancel{v} + v^2$$

$$v'x = 1 + v^2$$

$$\int \frac{dv}{1+v^2} = \int \frac{1}{x}\,dx \Rightarrow \tan^{-1}(v) = \ln x + C$$

$$\arctan(y/x) = \ln x + C$$

7. Solve the differential equation $y' = \frac{\sinh(x)}{3+4y}$, $y(0) = 1$. (12 points)

$$\frac{dy}{3+4y} = \int \sinh x \, dx$$

$$\frac{1}{4}\ln|3+4y| = \cosh x + C$$

$$\frac{1}{4}\ln|3+4| = 1 + C$$

$$\left(\frac{1}{4}\ln 7\right) - 1 = C$$

$$\frac{1}{4}\ln|3+4y| = \cosh x + \left(\frac{1}{4}\ln 7\right) - 1$$

8. Use the method of integrating factors to solve the differential equation $y' + \left(\frac{2}{t}\right)y = \frac{\sin t}{t^2}$, $y(\pi) = 0$. (12 points)

$$\mu = e^{\int 2/t \, dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$t^2 y' + 2t y = \sin t$$

$$\int (t^2 y)' = \int \sin t \, dt$$

$$t^2 y = -\cos t + C$$

$$y = \frac{-\cos t}{t^2} + \frac{C}{t^2}$$

$$0 = \frac{-(-1)}{\pi^2} + \frac{C}{\pi^2} \Rightarrow C = -1$$

$$y = \frac{-\cos t - 1}{t^2}$$

9. Convert the Bernoulli equation $y' + 5ty = y^{-2}$ into a first order linear differential equation. You do not have to solve. (8 points)

$$y'y^2 + 5ty^3 = 1$$
$$3y^2y' + 15ty^3 = 3$$

$$z = y^3$$
$$z' = 3y^2y'$$

$$z' + 15tz = 3$$

10. Use Euler's method to find the first three of 10 steps from $y(0) = 3$ to $y(1)$, under the differential equation $y' = \frac{4t^2+y}{\sqrt{t}-y}$. (12 points)

$$\frac{1-0}{10} = .1$$

$$m = \frac{0+3}{0-3} = -1$$

$$y(0+.1) = y(.1) = -1(0.1) + 3 = 2.9$$

$$m = \frac{4(.1)^2 + 2.9}{\sqrt{.1} - 2.9} = -1.137871195$$

$$y(.2) = -1.137871195(.1) + 2.9 = 2.78621288$$

$$m = \frac{4(.2)^2 + 2.786}{\sqrt{.2} - 2.786} = -1.259604011$$

$$y(.3) = -1.2596(.1) + 2.786 = 2.660252479$$

11. Classify the following differential equations as a) ordinary or partial, b) order, c) linear or nonlinear. (3 points each)

a. $t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \tanh y$ Ordinary, second order, nonlinear

b. $y' = \frac{3x^2-y}{y^2-7}$ nonlinear, first order, ordinary

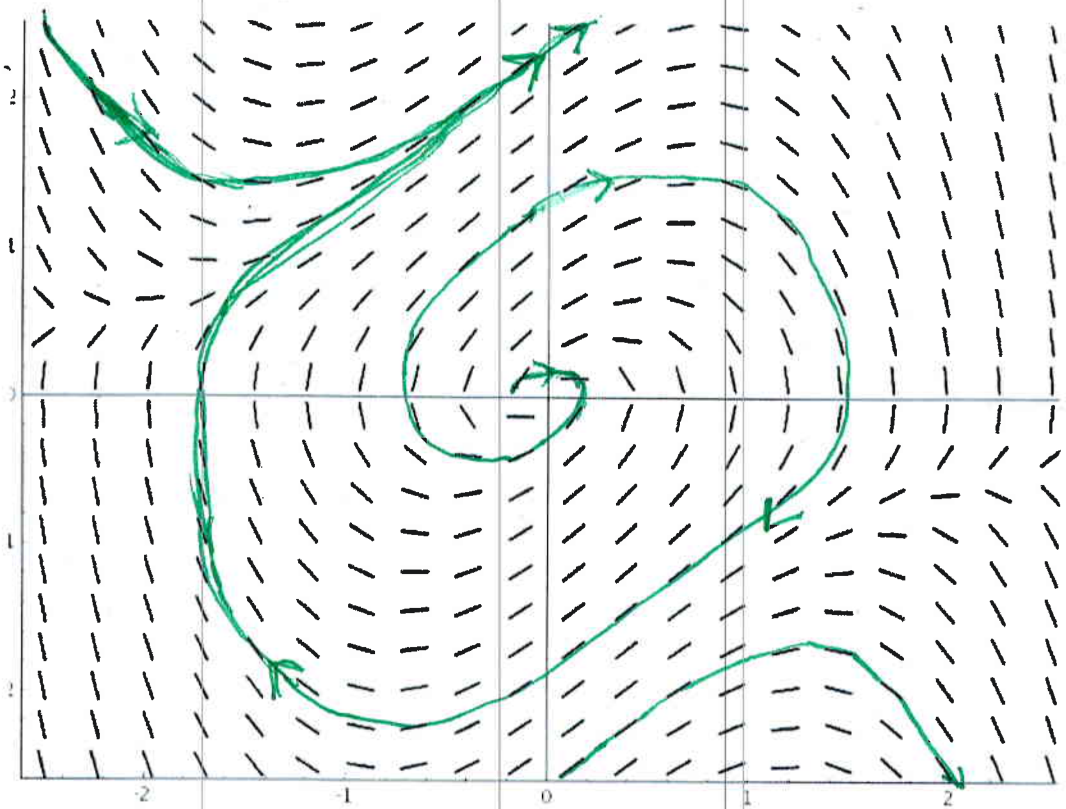
c. $u_{xxx} + u_y = e^{xy}$ linear, partial, 3rd order

12. Prove that the derivative of $\tanh x$ is $\operatorname{sech}^2 x$. (10 points)

$$\frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] = \frac{\cosh x \cdot \cosh - \sinh x \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} =$$

$$\frac{1}{\cosh^2 x} = \operatorname{Sech}^2 x.$$

13. Plot three distinct trajectories in the direction field shown below from three distinct initial conditions. (12 points)



14. Write the complex exponential $e^{(-1-2i)t}$ in standard form. (8 points)

$$e^{-t}e^{-2ti} = e^{-t}(\cos(-2t) + i\sin(-2t))$$

$$= e^{-t}(\cos(2t) - i\sin(2t))$$

$$= e^{-t}\cos 2t - e^{-t}\sin 2t$$