2415 Homework #5 Key

1. $C = \frac{1}{4}\times10^{-6}$ $R = 5000\,\Omega$ $L = 1$ $Q(0)=0,\ Q'(0)=0$

$$Q'' + 5000Q' + 4\times10^{6}Q = 12$$

$$r = \frac{-5000 \pm \sqrt{25\times10^{6} - 16\times10^{6}}}{2} = \frac{-5000 \pm \sqrt{9\times10^{6}}}{2} = \frac{-5000 \pm 3000}{2} = -4000, -1000$$

$$Q_c(t) = C_1 e^{-1000t} + C_2 e^{-4000t}$$

$Q_p(t) = A$ $Q'(t) = Q''(t) = 0$

$$A \times 4\times10^{6} = 12 \Rightarrow A = 3\times10^{-6}$$

$$Q(t) = C_1 e^{-1000t} + C_2 e^{-4000t} + 3\times10^{-6}$$

$$0 = C_1 + C_2 + 3\times10^{-6} \Rightarrow -3\times10^{-6} = C_1 + C_2$$

$$Q'(t) = -1000C_1 e^{-1000t} - 4000C_2 e^{-4000t}$$

$$0 = -1000C_1 - 4000C_2$$
$$-3\times10^{-3} = 1000C_1 + 1000C_2$$
$$> -3\times10^{-3} = -3000C_2 \Rightarrow C_2 = 10^{-6}$$

$$C_1 = -3\times10^{-6} - 10^{-6} = -4\times10^{-6}$$

$$Q(t) = -4\times10^{-6}e^{-1000t} + 10^{-6}e^{-4000t} + 3\times10^{-6}$$

$Q(.001) = 1.5\times10^{-6}$ $Q(.01) = 3\times10^{-6}$ $\lim_{t\to\infty} Q(t) = 3\times10^{-6}$

2. a. beats: no damping, period for forcing similar to homogeneous solution $y'' + 196y = \cos 15t$

b. resonance: no damping, period of forcing same as homogeneous

$$y'' + 196y = \cos 14t$$

c. asymptotically approaches zero → distinct roots, negative real part

$$y'' + 11y' + 10y = 0 \quad \text{or} \quad y'' + 11y' + 33y = 0$$

d. Contains a transient solution

$$y'' + 4y' + 3y = \cos 2t + e^{t}$$

e. oscillating steady state solution – forcing term w/ sine or cosine

$$y'' + 4y' + 3y = \sin 4t$$

1f. no damping $y'' + 196y = 0$

g. critical damping - repeated roots

$y'' + 2y' + y = 0$

3\. a. $\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x - \cos(n+m)x\,dx =$

$\frac{1}{2}\left[\frac{1}{n-m}\sin(n-m)x - \frac{1}{n+m}\sin(n+m)x\right|_{-\pi}^{\pi} = \frac{1}{2}\left[\frac{1}{n-m}\sin(n-m)\pi - \frac{1}{n+m}\sin(n+m)\pi\right.$

$\left. - \frac{1}{n-m}\sin(n-m)(-\pi) + \frac{1}{n+m}\sin(n+m)(-\pi)\right] = 0$

since n & m are integers and $\sin k\pi = 0$ for all $k \in \mathbb{Z}$

b. $\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x + \cos(n-m)x\,dx =$

$\frac{1}{2}\left[\frac{\sin(n+m)x}{n+m} + \frac{\sin(n-m)x}{n-m}\right|_{-\pi}^{\pi} = 0$ for the same reason as (a)

c. $\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x + \sin(n-m)x\,dx = 0$

since these are odd function on an interval $[-a, a]$.

4\. $f(x) = \begin{cases} -x & -2 \le x < 0 \\ x & 0 \le x < 2 \end{cases}$ $\quad f(x+4) = f(x)$

$S_m(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{m} \frac{\cos\left[\frac{(2n-1)\pi x}{2}\right]}{(2n-1)^2}$

See graphs on attached page.

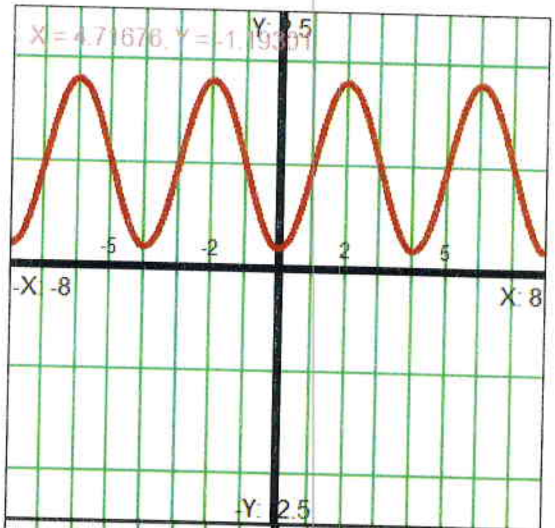
$S_1 = 1 - \frac{8}{\pi^2}\frac{\cos\left(\frac{\pi x}{2}\right)}{1}$ $\quad S_2 = 1 - \frac{8}{\pi^2}\frac{\cos\left(\frac{\pi x}{2}\right)}{1} - \frac{8}{\pi^2}\frac{\cos\left(\frac{3\pi x}{2}\right)}{9}$

$S_3 = 1 - \frac{8}{\pi^2}\left[\cos\left(\frac{\pi x}{2}\right) + \frac{\cos\left(\frac{3\pi x}{2}\right)}{9} + \frac{\cos\left(\frac{5\pi x}{2}\right)}{25}\right]$

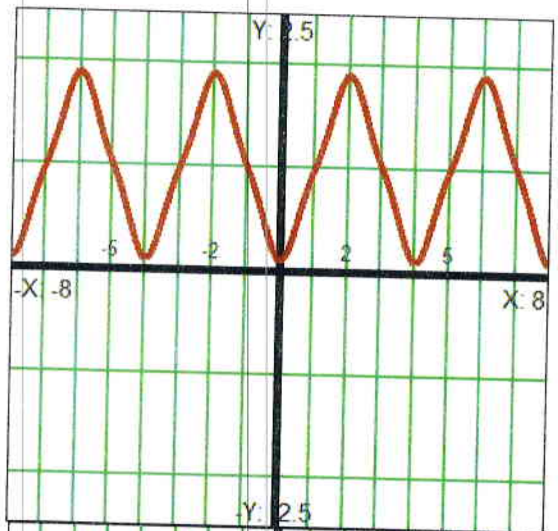
$S_4 = 1 - \frac{8}{\pi^2}\left[\cos\left(\frac{\pi x}{2}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{2}\right) + \frac{1}{49}\cos\left(\frac{7\pi x}{2}\right)\right]$

$S_6 = 1 - \frac{8}{\pi^2}\left[\cos\left(\frac{\pi x}{2}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{2}\right) + \frac{1}{49}\cos\left(\frac{7\pi x}{2}\right) + \frac{1}{81}\cos\left(\frac{9\pi x}{2}\right)\right.$
$\left. + \frac{1}{121}\cos\left(\frac{11\pi x}{2}\right)\right]$

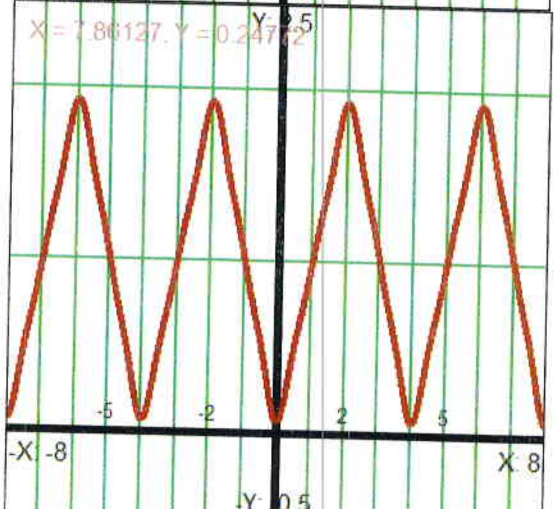
S_1



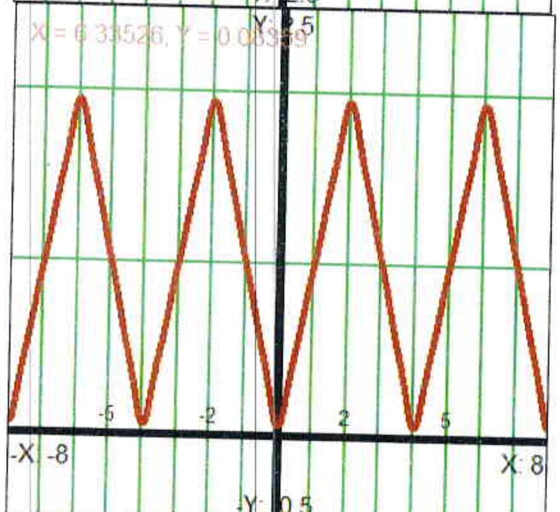
S_2



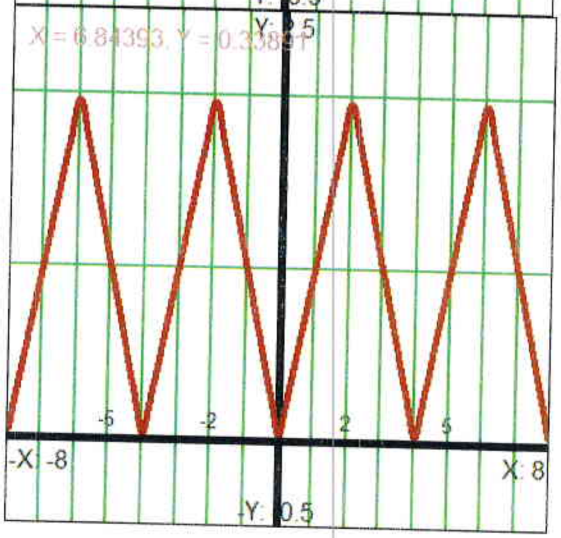
S_3



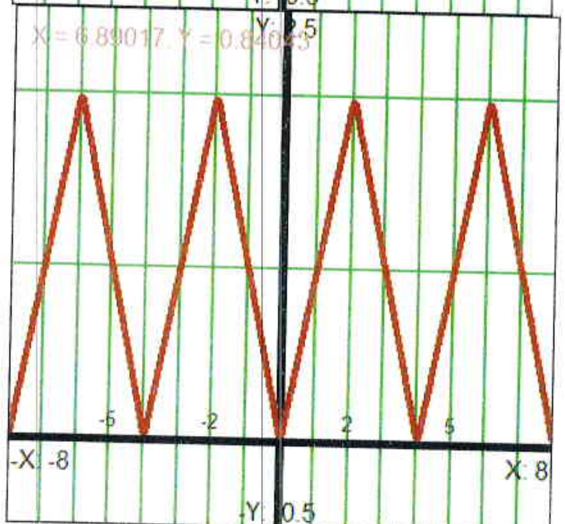
S_4

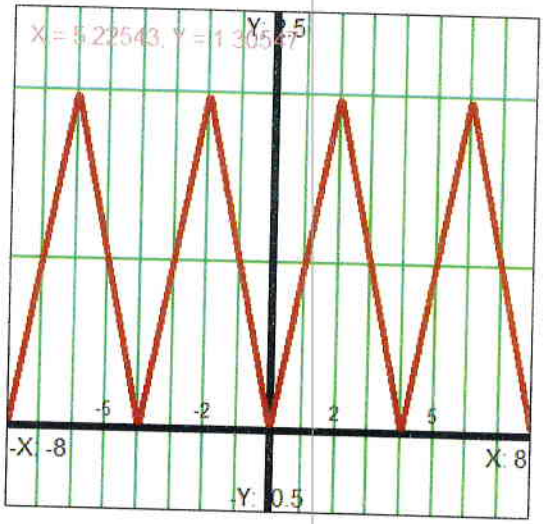


S_6



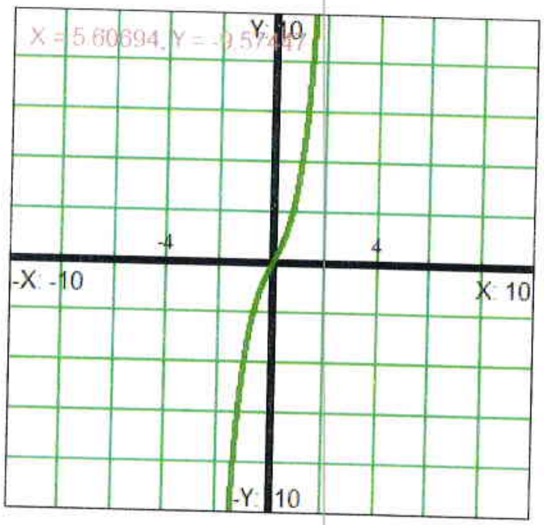
S_8

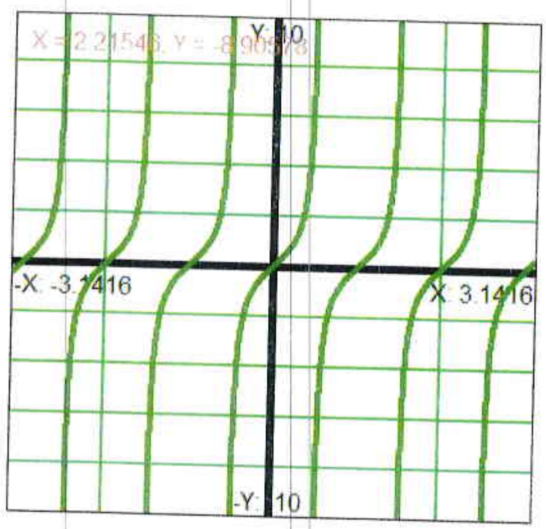



X = 5.22543, Y = 1.30547
Y: 2.5
-5
-2
2
5
-X: -8
X: 8
-Y: 0.5

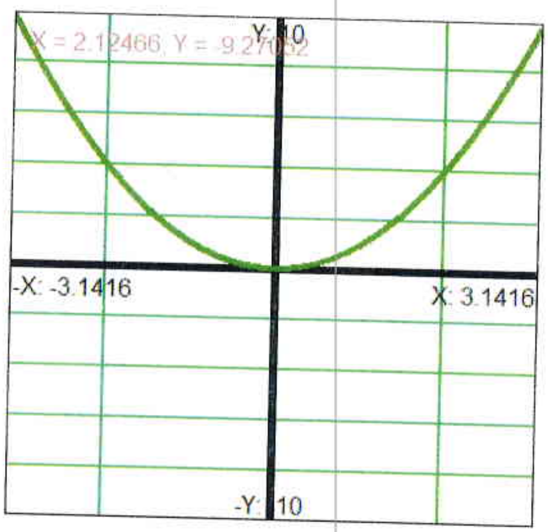
5d

5a.


X = 5.60694, Y = -9.57447
Y: 10
-4
4
-X: -10
X: 10
-Y: 10


X = 2.21546, Y = -8.90578
Y: 10
-X: -3.1416
X: 3.1416
-Y: 10

5b


X = 2.12466, Y = -9.27052
Y: 10
-X: -3.1416
X: 3.1416
-Y: 10

5c

4 con't.

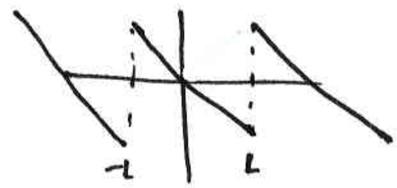
$$S_8 = 1 - \frac{8}{\pi^2}\left[\cos\left(\frac{\pi x}{2}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{2}\right) + \frac{1}{49}\cos\left(\frac{7\pi x}{2}\right) + \right.$$
$$\left. \frac{1}{81}\cos\left(\frac{9\pi x}{2}\right) + \frac{1}{121}\cos\left(\frac{11\pi x}{2}\right) + \frac{1}{169}\cos\left(\frac{13\pi x}{2}\right) + \frac{1}{225}\cos\left(\frac{15\pi x}{2}\right)\right]$$

$$S_{10} = S_8 + \frac{-1}{8\pi^2}\left[\frac{1}{289}\cos\left(\frac{17\pi x}{2}\right) + \frac{1}{361}\cos\left(\frac{19\pi x}{2}\right)\right]$$

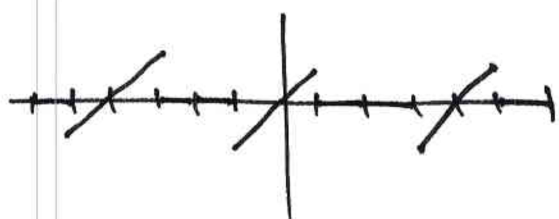
5. a. $\sinh(2x)$ not periodic, odd
   b. $\tan(\pi x)$ periodic (π), odd
   c. x^2 not periodic, even

See attached graphs

   d. $-x \quad -L \le x < L$ periodic $2L$
   $f(x+2L) = f(x)$ odd



   e. $\begin{cases} 0 & -2 \le x < -1 \\ x & -1 \le x < 1 \\ 0 & 1 \le x < 2 \end{cases}$ $f(x+4) = f(x)$ periodic odd



6. a. $a_n = \cos(n\pi) = (-1)^n$
   b. $\cos\left(\frac{n\pi}{2}\right) = b_n$ when n is even $n = 2k \Rightarrow \cos(k\pi) = (-1)^k$
   when n is odd $n = 2k+1 \Rightarrow \cos(2k+1)\pi = 0$
   $b_n = (-1)^{n/2}$ when n is even, 0 otherwise
   or $b_k = (-1)^k$

   c. $c_n = \sin\left(\frac{n\pi}{2}\right)$ when n is even $n = 2k$ $\sin(k\pi) = 0$.
   when n is odd $n = 2k+1$ $\sin\left(\frac{2k+1}{2}\right)\pi = (-1)^k$
   $c_k = (-1)^k$

   d. $d_n = \sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right) = \sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) = 2\sin\left(\frac{n\pi}{2}\right)$
   reduces as c does
   $d_k = 2(-1)^k$ for $n = 2k+1$ 0 otherwise (when n is even)

   e. $e_n = \cos\left(\frac{n\pi}{4}\right)$ 4 cases: $n = 4k, 4k+1, 4k+2, 4k+3$

6e cont'd

$n = 4k \quad \cos\left(\frac{4k\pi}{4}\right) = \cos(\pi k) = (-1)^k$

$n = 4k+1 \quad \cos\left(\frac{(4k+1)\pi}{4}\right) = \cos\left(k\pi + \frac{\pi}{4}\right) = (-1)^k\left(\frac{1}{\sqrt{2}}\right)$

$n = 4k+2 \quad \cos\left(\frac{(4k+2)\pi}{4}\right) = \cos\left(k\pi + \frac{\pi}{2}\right) = 0$

$n = 4k+3 \quad \cos\left(\frac{(4k+3)\pi}{4}\right) = \cos\left(k\pi + \frac{3\pi}{4}\right) = (-1)^{k+1}\left(\frac{1}{\sqrt{2}}\right)$

we would need to separate this into three different cases

$e_k = (-1)^k, \quad e_j = (-1)^j\left(\frac{1}{\sqrt{2}}\right) \quad \$ \quad e_m = (-1)^{m+1}\left(\frac{1}{\sqrt{2}}\right)$

$4k = n \qquad n = 4j+1 \qquad 4m+3 = n$

7. a. $a_0 = \frac{1}{L}\int_{-L}^{0} 1\,dx = \frac{1}{L}\left[x\right|_{-L}^{0} = \frac{L}{L} = 1$

$a_n = \frac{1}{L}\int_{-L}^{0} \cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} = \frac{1}{n\pi}\left[0 + \sin(n\pi)\right] = 0$

$b_n = \frac{1}{L}\int_{-L}^{0} \sin\left(\frac{n\pi x}{L}\right)dx = -\frac{1}{L}\cdot\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} = -\frac{1}{n\pi}\left[\cos(0) - \cos(n\pi)\right]$

$= -\frac{1}{n\pi}\left[1 - (-1)^n\right]$ n even $1 - 1 = 0$

n odd $1 - (-1) = 2$ $\qquad n = 2k+1$

$b_k = \frac{-2}{(2k+1)\pi}$

$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{-2}{(2k+1)\pi}\sin\left(\frac{(2k+1)\pi x}{L}\right) = f(x)$

b. $a_0 = \frac{1}{2}\int_{-2}^{0} -1\,dx + \frac{1}{2}\int_{0}^{2} 1\,dx = \frac{1}{2}\left[-x\Big|_{-2}^{0} + x\Big|_{0}^{2}\right] = \frac{1}{2}[2+2] = 2$

$a_n = \frac{1}{2}\left[\int_{-2}^{0} -\cos\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2}\cos\left(\frac{n\pi x}{2}\right)dx\right] = \frac{1}{2}\left[-\sin\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\Big|_{-2}^{0} + \right.$

$\left.\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_{0}^{2}\right] = \frac{1}{2}\left[0 + \sin(-n\pi)\cdot\frac{2}{n\pi} + \frac{2}{n\pi}\sin(n\pi) - \frac{2}{n\pi}\cdot 0\right] = 0$

$b_n = \frac{1}{2}\left[\int_{-2}^{0} -\sin\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2}\sin\left(\frac{n\pi x}{2}\right)dx\right] = \frac{1}{2}\left[\cos\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\Big|_{-2}^{0} + \right.$

7b cont'd

⑤

$$-\cos\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\Big|_0^2 = \frac{1}{2}\left[\frac{2}{n\pi}\cos(0) - \frac{2}{n\pi}\cos(-n\pi) - \frac{2}{n\pi}\cos(n\pi) + \cos(0)\cdot\frac{2}{n\pi}\right.$$

$$= \left[\frac{4}{n\pi} - \frac{4}{n\pi}(-1)^n\right]\frac{1}{2} = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

n is even $1-1=0$
n is odd $1+1=2$

for n odd $\Rightarrow$ $n = 2k+1 \Rightarrow \dfrac{2\cdot 2}{(2k+1)\pi} = \dfrac{4}{(2k+1)\pi}$

$$1 + \sum_{k=0} \frac{4}{(2k+1)\pi}\sin\left(\frac{(2k+1)\pi x}{2}\right) = g(x)$$

c. $a_0 = \dfrac{1}{2}\displaystyle\int_{-2}^{2}\frac{x^2}{2}\,dx = \frac{1}{2}\cdot\frac{1}{6}x^3\Big|_{-2}^{2} = \frac{1}{12}\left[8-(-8)\right] = \frac{16}{12} = \frac{4}{3}$

$a_n = \dfrac{1}{4}\displaystyle\int_{-2}^{2} x^2\cos\left(\frac{n\pi x}{2}\right)dx$

±	u	dv
+	x^2	$\cos\left(\frac{n\pi x}{2}\right)$
−	$2x$	$\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$
+	2	$\frac{-4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$
	0	$\frac{-8}{\pi^3 n^3}\sin\left(\frac{n\pi x}{2}\right)$

$$\frac{1}{4}\left[\frac{2x^2}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{8x}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) - \frac{16}{n^3\pi^3}\sin\left(\frac{n\pi x}{2}\right)\right|_{-2}^{2}$$

$$= \frac{1}{4}\left[\frac{8}{n\pi}\cancel{\sin(n\pi)}^{0} + \frac{16}{n^2\pi^2}\cos(n\pi) - \frac{16}{n^3\pi^3}\cancel{\sin(n\pi)}^{0} - \right.$$

$$\left. \frac{-8}{n\pi}\cancel{\sin(-n\pi)}^{0} + \frac{16}{n^2\pi^2}\cos(-n\pi) + \frac{16}{n^3\pi^3}\cancel{\sin(-n\pi)}^{0}\right] =$$

$$\frac{1}{4}\left[\frac{16}{n^2\pi^2}\right]\left[\cos(n\pi) + \cos(n\pi)\right] = \frac{4}{n^2\pi^2}\left[2(-1)^n\right] = \frac{8(-1)^n}{n^2\pi^2}$$

$b_n = \dfrac{1}{4}\displaystyle\int_{-2}^{2} x^2\sin\left(\frac{n\pi x}{2}\right) = 0$ odd function properties.

$$h(x) = \frac{2}{3} + \sum_{n=1}^{\infty}\frac{8(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$$