Math 2415, Quiz #12, Summer 2014 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Determine whether each of the problems can be solved by separation of variables.
   a. $xu_{xx} + u_t = 0$

   $XT = u \quad u_x = X'T \quad u_{xx} = X''T \quad u_t = XT'$

   $xX''T = -XT' \Rightarrow X'' = \frac{T'}{T} = \lambda$ yes

   b. $u_{xx} + (x+y)u_{yy} = 0$

   $X''Y + (x+y)XY'' = 0$

   $X''Y + x^2Y'' + XYY'' = 0$ no

   c. $u_{xx} + u_{yy} + xu = 0$

   $X''Y + XY'' + x^2Y = 0 \quad \frac{(X''+x^2)\cancel{Y}}{X} = \frac{\cancel{X}Y''}{Y}$ yes

2. Solve the heat conduction equation $u_{xx} = u_t, u(0,t) = 0, u(40,t) = 0, u(x,0) = \begin{cases} x, & 0 \le x < 20 \\ 40 - x, & 20 \le x \le 40 \end{cases}$.

$\alpha = 1$

$L = 40$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} c_n e^{-\frac{n\pi^2 t}{40}} \sin\frac{n\pi x}{40}$$

$$c_n = \frac{2}{40}\int_0^{40} f(x)\sin\frac{n\pi x}{40}\,dx = \frac{1}{20}\left[\int_0^{20} x\sin\left(\frac{n\pi x}{40}\right)dx + \int_{20}^{40}(40-x)\sin\frac{n\pi x}{40}dx\right]$$

$$= \frac{1}{20}\left[-x\cos\left(\frac{n\pi x}{40}\right)\cdot\frac{40}{n\pi} + \left(\frac{40}{n\pi}\right)^2\sin\left(\frac{n\pi x}{40}\right)\Big|_0^{20} - 40\frac{40}{n\pi}\cos\left(\frac{n\pi x}{40}\right)\Big|_{20}^{40} - \int_{20}^{40} x\sin\left(\frac{n\pi x}{40}\right)dx\right.$$

$$\frac{1}{20}\left[\frac{-20\cdot 40}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{40}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) + 0 - 0 - \frac{40^2}{n\pi}\cos n\pi + \frac{40^2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \right.$$

$$\left(-x\cos\left(\frac{n\pi x}{40}\right)\left(\frac{40}{n\pi}\right) + \left(\frac{40}{n\pi}\right)^2\sin\left(\frac{n\pi x}{40}\right)\right)\Big|_{20}^{40}$$

$$= \frac{1}{20}\left[\cancel{\frac{-20\cdot 40}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \left(\frac{40}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) - \cancel{\frac{40^2}{n\pi}(-1)^n} + \cancel{\frac{40^2}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \cancel{\frac{40^2}{n\pi}\underbrace{\cos(n\pi)}_{(-1)^n}} - \cancel{\left(\frac{40}{n\pi}\right)^2\sin n\pi}\right.$$

$$\left. - \cancel{\frac{20\cdot 40}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \left(\frac{40}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{1}{20}\left[2\left(\frac{40}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right]$$

n even = 0
n odd = ±1

$$\sum_{k=1}^{\infty} c_k e^{-\frac{(2k+1)\pi^2 t}{40}} \sin\left(\frac{(2k+1)\pi x}{40}\right)$$

$$c_k = \frac{1}{\cancel{10}} \cdot \frac{40 \cdot 4\cancel{0}}{n^2\pi^2} (-1)^k = (-1)^k \frac{160}{n^2\pi^2}$$

$$u(x,t) = \sum_{k=1}^{\infty} (-1)^k \frac{160}{n^2\pi^2} e^{-\frac{(2k+1)\pi^2 t}{40}} \sin\left(\frac{(2k+1)\pi x}{40}\right)$$