Math 2415, Exam #2, Summer 2015 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Describe when a 'resonance' phenomenon occurs in a forced spring problem. (6 points)

in an undamped system when the frequency of the forcing function is the same as the natural frequency of the system.

2. Suppose that a mass of 10 kg stretches a spring 2 cm. Suppose that the mass is attached to a viscous damper with a damping constant of 100 Ns/m. If the mass is pulled down an additional four centimeters and then released, find the differential equation and initial conditions to be used to solve for the position of the system. (You do not need to solve the equation, just set it up.) (10 points)

$m = 10$ $\quad F = 10 \cdot 9.8 = k \cdot .02 \Rightarrow k = \frac{98}{.02} = 4900$

$\gamma = 100$ $\quad y(0) = -.04 \quad y'(0) = 0$

$$my'' + \gamma y' + ky = 0$$

$$\boxed{\begin{array}{l} 10y'' + 100y' + 4900y = 0 \\ y(0) = -.04 \\ y'(0) = 0 \end{array}}$$

or $y'' + 10y' + 490y = 0$

3. For each of the proposed solutions to a spring problem, describe any salient characteristics of the system. For instance, what is the transient solution (if it exists)? What is the steady state solution (if it exists)? Is the system undamped, underdamped, critically damped or overdamped? What is the frequency (or quasi-frequency) of the system? Does the system experience resonance or beats? [Hint: it may help to graph the system.] (8 points each)

a. $y(t) = \underbrace{e^{-t/2}\sin 5t + 3e^{-t/2}\cos 5t}_{\text{transient}} + \underbrace{\sin 2t}_{\text{steady state}}$

underdamped

quasi-frequency is 5

no resonance or beats

b. $y(t) = \underbrace{e^{-0.1t} - te^{-0.1t}}_{\text{transient solution}} + \underbrace{\cos(\pi t)}$ ← steady state

critically damped

no frequency for system

no resonance or beats

c. $y(t) = 9\sin\left(\frac{\sqrt{3}}{2}t\right) - 5\cos\left(\frac{\sqrt{3}}{2}t\right) + t\cos\left(\frac{\sqrt{3}}{2}t\right)$

all terms are part of the steady state, therefore, system is undamped

the system experiences resonance since natural frequency is $\frac{\sqrt{3}}{2}$ and so, it seems, is the forcing term.

4. Find the eigenvalues and eigenfunctions for the differential equation $y'' + \lambda y = 0, y'(0) = 0$, $y'(L) = 0$. Be sure to check all three cases ($\lambda < 0, \lambda = 0, \lambda > 0$). Determine if the solution in each case exists, is trivial, and/or is unique. (18 points)

$r^2 + \lambda = 0$

$r = \pm\sqrt{-\lambda}$

① for $\lambda = 0$

repeated root $r = 0$

$y(t) = At + B$

$y'(t) = A$

if $y'(0) = 0 \Rightarrow A = 0$

$y'(L) = 0$ satisfied

$y = B$ is an eigenfunction (i.e. y = constant or 1) $\lambda = 0$ eigenvalue

② for $\lambda > 0$ let $\lambda = \mu^2$

$r = \pm\sqrt{-\mu^2} = \pm\mu i$

$y(t) = A\cos\mu t + B\sin\mu t$

$y'(t) = -A\mu\sin\mu t + B\mu\cos\mu t$

$y'(0) = 0 \Rightarrow B = 0$

$y'(L) = 0 \Rightarrow -A\mu\sin(\mu L) = 0$

$A \neq 0$ if $\mu L = n\pi$

$\mu = \frac{n\pi}{L}$

$\lambda = \frac{n^2\pi^2}{L^2}$ eigenvalue
eigenfunction $y(t) = A\cos\mu t$

③ $\lambda < 0$ let $\lambda = -\mu^2$

$r = \pm\sqrt{\mu^2} = \pm\mu$

$y(t) = Ae^{\mu t} + Be^{-\mu t}$

$y'(t) = A\mu e^{\mu t} - B\mu e^{-\mu t}$

$y'(0) = 0 \Rightarrow$

$A\mu - B\mu = 0 \Rightarrow$

$A = B$

$y'(L) = 0 \Rightarrow$

$A\mu e^{\mu L} - A\mu e^{-\mu L} = 0$

$A\mu(e^{\mu L} - e^{-\mu L}) = 0$

$\mu = 0$

but accounted for in case #1

$\Rightarrow$ trivial solutions only.

5. Give examples of **two** functions with a fixed period (and state the period for each), and **two** functions that are fundamentally non-periodic. (8 points)

periodic

$\sin(2t)$, $\tan(3t)$

period $= \frac{2\pi}{2} = \pi$ period $= \frac{\pi}{3}$

nonperiodic

x^2

x^3

6. Find the Fourier series for the function $f(x) = \begin{cases} 3x, & -2 \le x < 0 \\ 1, & 0 \le x < 2 \end{cases}$, $f(x+4) = f(x)$. Be sure to simplify the coefficients as much as possible. (20 points)

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} 3x\,dx + \int_{0}^{2} 1\,dx\right] =$$

$$\frac{1}{2}\left[\frac{3}{2}x^2 \Big|_{-2}^{0} + x\Big|_0^2\right] = \frac{1}{2}\left[-\frac{3}{\cancel{2}}(\overset{2}{\cancel{4}}) + \quad + 2\right] =$$

$$\frac{1}{2}\left[-6 + 2\right] = \frac{1}{2}\left[-4\right] = -2$$

$$a_n = \frac{1}{2}\left[\int_{-2}^{0}(3x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_0^2 1\cos\left(\frac{n\pi x}{2}\right)dx\right] =$$

$$\frac{1}{2}\left[\cancel{\frac{(3x)\cdot 2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)}^{\to 0} - \frac{3\cdot 2}{n\pi}\left(\frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right)\Big|_{-2}^{0} + \cancel{\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)}^{\to 0}\Big|_0^2\right.$$

$+$	$3x$	$\cos\left(\frac{n\pi x}{2}\right)$
$-$	3	$\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$
$+$	0	$\frac{-4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$

$$\frac{1}{2}\left[\frac{12}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{0}\right] = \frac{6}{n^2\pi^2}\left((1) - \underbrace{\cos(-n\pi)}_{(-1)^n}\right) = \frac{6}{n^2\pi^2}(1-(-1)^n)$$

n even $\Rightarrow 0$
n odd $\Rightarrow 2$

$$\frac{12}{(2k+1)^2\pi^2} = a_k$$

$$b_n = \frac{1}{2}\left[\int_{-2}^{0}(3x)\sin\left(\frac{n\pi x}{2}\right)dx + \int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx\right] =$$

$+$	$3x$	$\sin\left(\frac{n\pi x}{2}\right)$
$-$	3	$-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$
$+$	0	$\frac{-4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)$

$$= \frac{1}{2}\left[-\frac{(3x)2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \cancel{\frac{12}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)}^{\to 0}\Big|_{-2}^{0} + -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2\right. =$$

$$= \frac{1}{2}\left[\frac{0}{n\pi}(1) - \frac{12}{n\pi}\underbrace{(-1)^n}_{\cos(-n\pi)} - \frac{2}{n\pi}\underbrace{(\cos n\pi)}_{(-1)^n} + \frac{2}{n\pi}(1)\right] =$$

$$= \frac{1}{2}\left[\frac{2}{n\pi} - \frac{14}{n\pi}(-1)^n\right] = \frac{1}{n\pi}\left[1 - 7(-1)^n\right]$$

$\Rightarrow$ n even $\quad -6 \quad 2k$
n odd $\quad 8 \quad (2k+1)$

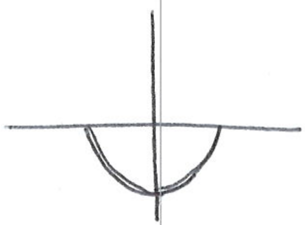
$$b_k = \frac{-6}{n\pi} \text{ for } \sin\left(\frac{\cancel{2}k\pi x}{\cancel{2}}\right) \quad \text{and} \quad \frac{8}{n\pi} \text{ for } \sin\left(\frac{(2k+1)\pi x}{2}\right)$$

$$f(x) = -1 + \sum_{k=1}^{\infty}\left[\frac{12}{(2k+1)^2\pi^2}\cos\left(\frac{(2k+1)\pi x}{2}\right) - \frac{6}{n\pi}\sin(k\pi x) + \frac{8}{n\pi}\sin\left(\frac{(2k+1)\pi x}{2}\right)\right]$$

7. For each of the functions below, rewrite the function so that the resulting function is a) even, b) odd. Sketch the graph in each case. State the length of the period. (10 points each)

i. $f(x) = x^2 - 1, \ 0 \le x \le 1$

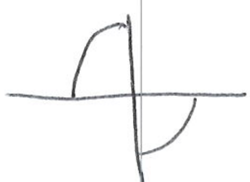
a)
even



$f(x) = x^2 - 1 \quad -1 \le x \le 1$

period $= 2$

b)
odd



$$f(x) = \begin{cases} 1 - x^2 & -1 \le x < 0 \\ x^2 - 1 & 0 \le x \le 1 \end{cases}$$

period $= 2$

ii. $f(x) = \begin{cases} 2, & 0 \le x < 2 \\ 3 - x, & 2 \le x < 3 \end{cases}$

a)
even

$$f(x) = \begin{cases} x + 3 & -3 \le x < -2 \\ 2 & -2 \le x < 2 \\ 3 - x & 2 \le x < 3 \end{cases}$$

period $= 6$

b)
odd

$$f(x) = \begin{cases} -x - 3 & -3 \le x < -2 \\ -2 & -2 \le x < 0 \\ 2 & 0 \le x < 2 \\ 3 - x & 2 \le x < 3 \end{cases}$$

period $= 6$

8. Under what conditions does the Fourier series contain only sine functions? Under what conditions does it contain only cosine function? When must it contain both types of functions? Explain your reasoning. (6 points)

Only sine functions when the underlying function being approximated is odd (origin symmetry)

it will contain only cosine functions when the underlying function being approximated is even (y-axis symmetry)

it must have both in the absence of either symmetry

9. Determine if the partial differential equations below can be solved with the method of separation of variables. (6 points each)

a. $t^2 u_{xx} + u_t = 0$

$u = XT$

$u_{xx} = X''T \quad u_t = XT'$

$t^2 X''T = -XT'$

$\frac{X''}{X} = \frac{-T'}{t^2 T}$

$\boxed{\text{yes}}$

b. $u_{xx} + x u_{xt} + u_{tt} = 0$

$u = XT \quad u_{xx} = X''T \quad u_{xt} = X'T' \quad u_{tt} = XT''$

$X''T + xX'T' + XT'' = 0$

$\boxed{\text{no}}$

c. $t^2 u_{xx} + x^3 u_t = 0$

$u = XT \quad u_{xx} = X''T \quad u_t = XT'$

$t^2 X''T = -x^3 XT'$

$\frac{X''}{x^3 X} = \frac{-T'}{t^2 T}$

$\boxed{\text{yes}}$

10. The solution to the heat equation is given by $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$ where $c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$. Find the solution for the case when the rod is 50 cm long and both ends are maintained at 0°C for all $t > 0$. Suppose the initial temperature distribution is given by $u(x,0) = \begin{cases} 0, & 0 \le x < 15 \\ 25, & 15 \le x < 35 \\ 0, & 35 \le x \le 50 \end{cases}$. Assume that $\alpha^2 = 1$. (20 points)

$$C_n = \frac{2}{50}\int_0^{50} u(x,0)\sin\left(\frac{n\pi x}{50}\right)dx = \frac{1}{\cancel{25}}\int_{15}^{35} \cancel{25}\sin\left(\frac{n\pi x}{50}\right)dx =$$

$$-\frac{50}{n\pi}\cos\left(\frac{n\pi x}{50}\right)\Big|_{15}^{35} = \frac{-50}{n\pi}\left[\cos\left(\frac{n\pi\cdot 35}{50}\right) - \cos\left(\frac{n\pi\cdot 15}{50}\right)\right]$$

$$= \frac{-50}{n\pi}\left(\cos\left(\frac{7n\pi}{10}\right) - \cos\left(\frac{3n\pi}{10}\right)\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{50}{n\pi}\left(\cos\left(\frac{3n\pi}{10}\right) - \cos\left(\frac{7n\pi}{10}\right)\right)e^{-\frac{n^2\pi^2 t}{50^2}}\sin\left(\frac{n\pi x}{50}\right)$$

11. Consider a bar 60 cm long that is made of a material for which $\alpha^2 = 2$ and whose ends are insulated. Suppose that the initial temperature is zero except for the interval $25 < x < 50$, where the initial temperature is 185°C. Write the set of equations and initial conditions for the problem with proper notation. You do not need to solve. (6 points)

$$2u_{xx} = u_t$$

$$u(x,0) = \begin{cases} 0 & 0 \le x < 25 \\ 185 & 25 \le x \le 50 \\ 0 & 50 < x < 60 \end{cases}$$

$$u_x(0,t) = u_x(l,t) = 0$$

12. Suppose that the solution to a Fourier series is given by $f(x) \approx \sum_{n=1}^{N} -\frac{2(-1)^n}{n\pi}\sin(n\pi x)$. Sketch the graph of the function using $N = 1, N = 2, N = 5$. [Hint: you will need 3 graphs, and you may use your calculator to obtain them. Recall that your second and third graphs are a sum of terms, not a single term.] (20 points)

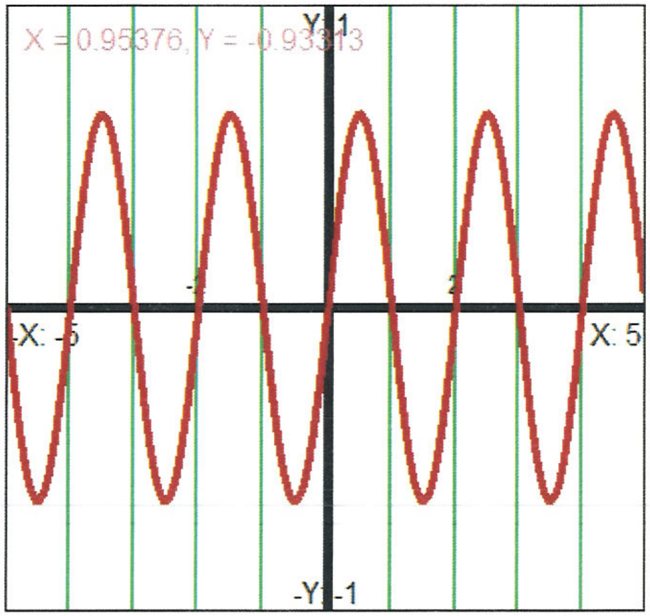
$N=1$ $\quad -\frac{2(-1)^1}{1\pi}\sin(\pi x) = \frac{2}{\pi}\sin \pi x$

$N=2$ $\quad \frac{2}{\pi}\sin(\pi x) - \frac{\cancel{2}(-1)^2}{\cancel{2}\pi}\sin(2\pi x) = \frac{2}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x)$

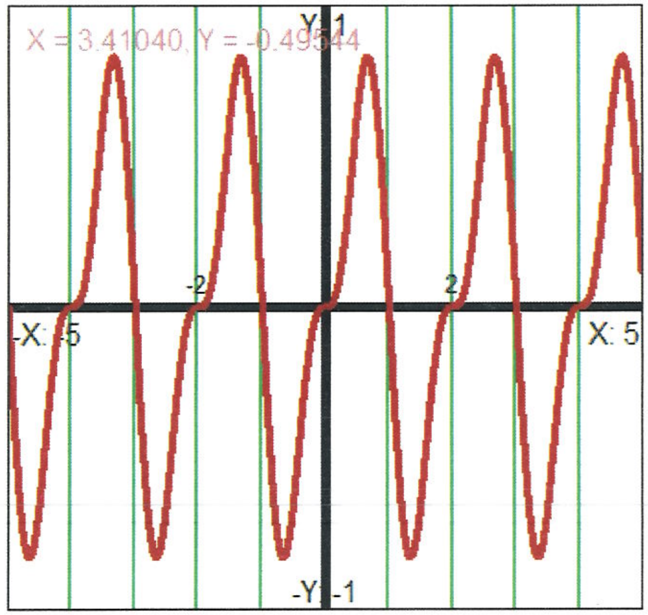
$N=5$ $\quad \frac{2}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x) - \frac{2(-1)^3}{3\pi}\sin(3\pi x) - \frac{2(-1)^4}{4\pi}\sin(4\pi x) - \frac{2(-1)^5}{5\pi}\sin(5\pi x)$

$= \frac{2}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x) + \frac{2}{3\pi}\sin(3\pi x) - \frac{1}{2\pi}\sin(4\pi x) + \frac{2}{5\pi}\sin(5\pi x)$

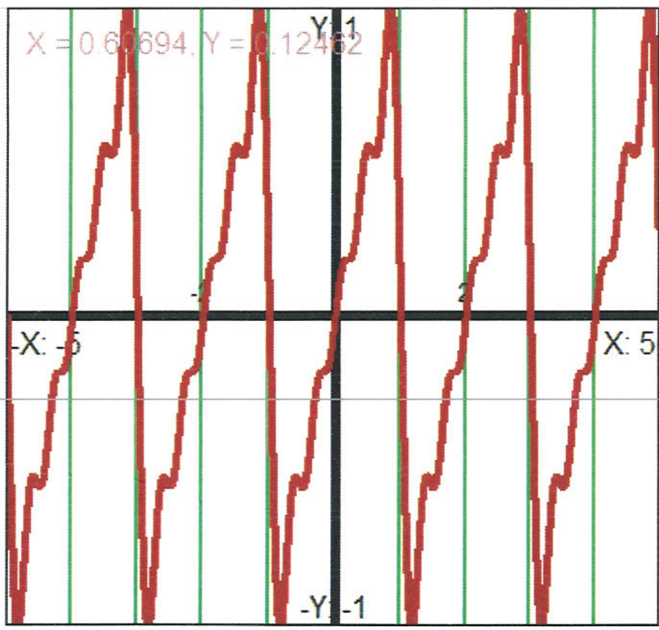
graph attached on next page


X = 0.95376, Y = -0.93313
Y: 1
-X: -5
X: 5
-Y: -1

N=1


X = 3.41040, Y = -0.49544
Y: 1
-X: -5
-2
2
X: 5
-Y: -1

N=2


X = 0.60694, Y = 0.12462
Y: 1
-X: -5
2
X: 5
-Y: -1

N=5