Linear ODEs KEY

1. $\vec{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \vec{x}$

$$\begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda)+6 = \lambda^2+3\lambda-4+6 = \lambda^2+3\lambda+2 =$$

$$(\lambda+2)(\lambda+1)=0 \qquad \lambda=-2,-1$$

$\lambda_1=-2$ $\begin{bmatrix} 1-(-2) & -2 \\ 3 & -4-(-2) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}$ $\quad 3x_1 - 2x_2 = 0$

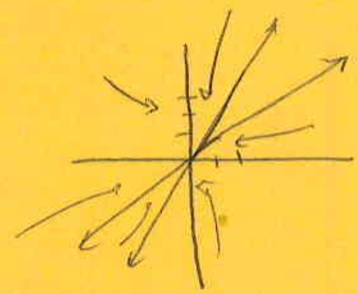
$\Rightarrow \begin{matrix} x_1 = \frac{2}{3}x_2 \\ x_2 = x_2 \end{matrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\lambda_2=-1$ $\begin{bmatrix} 1-(-1) & -2 \\ 3 & -4(-1) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 - x_2 = 0$

$\Rightarrow \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

origin is attracting



2. $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}$

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda)+3 = \lambda^2-4+3 = \lambda^2-1=0 \qquad \lambda=\pm 1$$

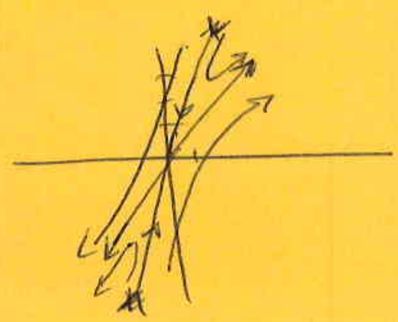
$\lambda_1=1$ $\begin{bmatrix} 2-1 & -1 \\ 3 & -2-1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 - x_2 = 0$

$\Rightarrow \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2=-1$ $\begin{bmatrix} 2-(-1) & -1 \\ 3 & -2-(-1) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad 3x_1 - x_2 = 0$

$\Rightarrow \begin{matrix} x_1 = \frac{1}{3}x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$



origin is a saddle point

3. $\vec{x} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x}$

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)(-2-\lambda) - 1 = \lambda^2 + 4\lambda + 4 - 1 = \lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+3)(\lambda+1) = 0 \qquad \lambda = -3, -1$$

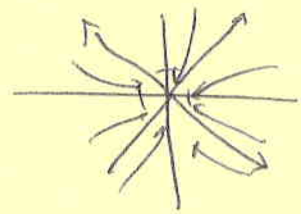
$\lambda_1 = -3$ $\begin{bmatrix} -2-(-3) & 1 \\ 1 & -2-(-3) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $x_1 + x_2 = 0$

$\Rightarrow$ $x_1 = -x_2$, $x_2 = x_2$ $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$ $\begin{bmatrix} -2-(-1) & 1 \\ 1 & -2-(-1) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ $-x_1 + x_2 = 0$

$\Rightarrow$ $x_1 = x_2$, $x_2 = x_2$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{x} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$



origin is attracting

4. $\vec{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \vec{x}$

$$\begin{vmatrix} 4-\lambda & -3 \\ 8 & -6-\lambda \end{vmatrix} = (4-\lambda)(-6-\lambda) + 24 = \lambda^2 + 2\lambda - 24 + 24 = \lambda^2 + 2\lambda$$

$$\lambda(\lambda+2) = 0 \qquad \lambda = 0, \lambda = -2$$

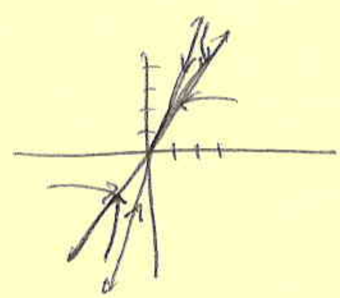
$\lambda_1 = 0$ $\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}$ $4x_1 - 3x_2 = 0$

$\Rightarrow$ $x_1 = \frac{3}{4} x_2$, $x_2 = x_2$ $\Rightarrow \vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\lambda_2 = -2$ $\begin{bmatrix} 4-(-2) & -3 \\ 8 & -6-(-2) \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ $2x_1 - x_2 = 0$

$\Rightarrow$ $x_1 = \frac{x_2}{2}$, $x_2 = x_2$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\vec{x} = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$$



$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is stable, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ attracting

5. $\vec{x}' = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix} \vec{x}$

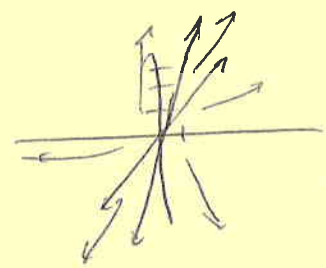
$$\begin{vmatrix} 7-\lambda & -1 \\ 3 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda) + 3 = \lambda^2 - 10\lambda + 21 + 3 = \lambda^2 - 10\lambda + 24 = 0$$

$$(\lambda-4)(\lambda-6) = 0 \qquad \lambda = 4, 6$$

$\lambda_1 = 4 \quad \begin{bmatrix} 7-4 & -1 \\ 3 & 3-4 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad 3x_1 - x_2 = 0 \Rightarrow x_1 = \frac{1}{3}x_2, \; x_2 = x_2 \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda_2 = 6 \quad \begin{bmatrix} 7-6 & -1 \\ 3 & 3-6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 - x_2 = 0 \Rightarrow x_1 = x_2, \; x_2 = x_2 \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$



origin is repelling

6. $\vec{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} \vec{x}$

$$\begin{vmatrix} 4-\lambda & -3 \\ 6 & -2-\lambda \end{vmatrix} = (4-\lambda)(-2-\lambda) + 18 = \lambda^2 - 2\lambda - 8 + 18 = \lambda^2 - 2\lambda + 10 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

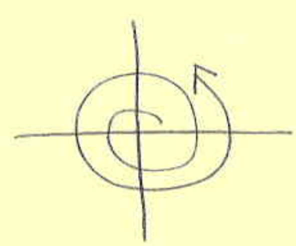
$\lambda_1 = \quad \begin{bmatrix} 4-(1+3i) & -3 \\ 6 & -2-(1+3i) \end{bmatrix} = \begin{bmatrix} 3-3i & -3 \\ 6 & -3-3i \end{bmatrix} \Rightarrow \begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix}$ $\quad 2x_1 + (-1-i)x_2 = 0 \Rightarrow x_1 = \frac{(1+i)}{2}x_2, \; x_2 = x_2$

$$\vec{v} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{(1+3i)t} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{t} e^{3ti} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{t}(\cos 3t + i \sin 3t)$$

$$e^t \begin{bmatrix} \cos 3t + i\cos 3t + i\sin 3t - \sin 3t \\ 2\cos 3t + 2i\sin 3t \end{bmatrix}$$

$$\vec{x} = c_1 \begin{bmatrix} \cos 3t - \sin 3t \\ 2\cos 3t \end{bmatrix} e^t + c_2 \begin{bmatrix} \cos 3t + \sin 3t \\ 2\sin 3t \end{bmatrix} e^t$$



Spirals out
(origin repels)

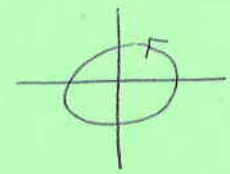
7. $\vec{x}' = \begin{bmatrix} -2 & 1 \\ -8 & 2 \end{bmatrix} \vec{x}$

$$\begin{vmatrix} -2-\lambda & 1 \\ -8 & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda) + 8 = \lambda^2 - 4 + 8 = \lambda^2 + 4 = 0 \qquad \lambda = \pm 2i$$

$$\begin{bmatrix} -2-2i & 1 \\ -8 & 2-2i \end{bmatrix} \Rightarrow \begin{bmatrix} -2-2i & 1 \\ -4 & 1-i \end{bmatrix} \qquad \begin{aligned} &-4x_1 + (1-i)x_2 = 0 \\ \Rightarrow\ & x_1 = \tfrac{1-i}{4} x_2 \\ & x_2 = x_2 \end{aligned} \qquad \vec{v} = \begin{bmatrix} 1-i \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1-i \\ 4 \end{bmatrix} e^{2it} = \begin{bmatrix} 1-i \\ 4 \end{bmatrix} (\cos 2t + i \sin 2t) = \begin{bmatrix} \cos 2t - i\cos 2t + i \sin 2t + \sin 2t \\ 4\cos 2t + 4i \sin 2t \end{bmatrix}$$

$$\vec{x} = c_1 \begin{bmatrix} \cos 2t + \sin 2t \\ 4\cos 2t \end{bmatrix} + c_2 \begin{bmatrix} \cos 2t + \sin 2t \\ 4 \sin 2t \end{bmatrix}$$



Stable ellipse

8. $\vec{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \vec{x}$

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} \Rightarrow (3-\lambda)(-1-\lambda) + 8 = \lambda^2 - 2\lambda - 3 + 8 = \lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

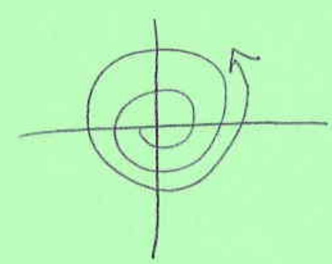
$$\begin{bmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{bmatrix} \Rightarrow \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \Rightarrow \begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix} \qquad \begin{aligned} &2x_1 + (-1-i)x_2 = 0 \\ \Rightarrow\ & x_1 = \tfrac{1+i}{2} x_2 \\ & x_2 = x_2 \end{aligned}$$

$$\vec{v} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{(1+2i)t} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{t} (\cos 2t + i \sin 2t) =$$

$$e^t \begin{bmatrix} \cos 2t + i \sin 2t + i\cos 2t - \sin 2t \\ 2\cos 2t + 2i \sin 2t \end{bmatrix}$$

$$\vec{x} = c_1 \begin{bmatrix} \cos 2t - \sin 2t \\ 2\cos 2t \end{bmatrix} e^t + c_2 \begin{bmatrix} \sin 2t + \cos 2t \\ 2 \sin 2t \end{bmatrix} e^t$$



Spirals out
(origin repels)

9. $\vec{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \vec{x}$

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda)+5 = \lambda^2+2\lambda-3+5 = \lambda^2+2\lambda+2$$

$$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\begin{bmatrix} 1-(-1+i) & -5 \\ 1 & -3-(-1+i) \end{bmatrix} = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix}$$

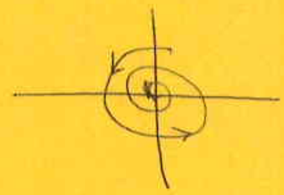
$x_1 + (-2-i)x_2 = 0$

$\Rightarrow \quad x_1 = (2+i)x_2 \qquad \vec{v} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$

$x_2 = x_2$

$$\begin{bmatrix} 2+i \\ 1 \end{bmatrix} e^{(-1+i)t} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} e^{-t}(\cos t + i\sin t) = e^{-t}\begin{bmatrix} 2\cos t + i\cos t + 2i\sin t - \sin t \\ \cos t + i\sin t \end{bmatrix}$$

$$\vec{x} = c_1 \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix} e^{-t}$$



Origin attracts / spirals in

10. $\vec{x}' = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \vec{x}$

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix}$$

$= (3-\lambda)[-\lambda(3-\lambda)-4] - 2[2(3-\lambda)-8] + 4[4+4\lambda] =$

$(3-\lambda)[\lambda^2-3\lambda-4] - 2[6-2\lambda-8] + 16+16\lambda =$

$3\lambda^2 - \lambda^3 - 9\lambda + 3\lambda^2 - 12 + 4\lambda - 12 + 4\lambda + 16 + 16 + 16\lambda =$

$-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0 \Rightarrow \lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$

$$\begin{array}{r} \lambda^2 - 7\lambda - 8 \\ \lambda+1 \enclose{longdiv}{\lambda^3 - 6\lambda^2 - 15\lambda - 8} \\ -\lambda^3 \mp \lambda^2 \\ \hline -7\lambda^2 - 15\lambda - 8 \\ +7\lambda^2 + 7\lambda \\ \hline -8\lambda - 8 \\ -8\lambda - 8 \\ \hline 0 \end{array}$$

$(\lambda+1)(\lambda^2-7\lambda-8) = (\lambda+1)(\lambda+1)(\lambda-8) = 0$

$\lambda = -1, \lambda = +8$

$\lambda_1 = -1 \quad \begin{bmatrix} 3+1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$2x_1 + x_2 + 2x_3 = 0$

$\Rightarrow \quad x_1 = -\frac{1}{2}x_2 - x_3 \qquad \vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$x_2 = x_2$

$x_3 = x_3$

$\lambda_3 = +8 \quad \begin{bmatrix} 3-8 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & 3-8 \end{bmatrix} \Rightarrow \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - x_3 = 0$

$x_2 - \frac{1}{2}x_3 = 0$

$\Rightarrow \quad x_1 = x_3 \qquad \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$x_2 = \frac{1}{2}x_3$

$x_3 = x_3$

$$\vec{x} = c_1 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} te^{-t} + c_3 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} e^{8t}$$

Saddle point

11. $\vec{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \vec{x}$ $\quad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)(1-\lambda)+4\right]$

$= (1-\lambda)\left[\lambda^2 - 2\lambda + 1 + 4\right] = (1-\lambda)(\lambda^2 - 2\lambda + 5)$ $\quad \lambda = 1$

$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$\lambda = 1 \quad \begin{bmatrix} 1-1 & 0 & 0 \\ 2 & 1-1 & -2 \\ 3 & 2 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1.5 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \begin{matrix} x_1 - x_3 = 0 \\ x_2 + 1.5x_3 = 0 \end{matrix} \Rightarrow \quad \begin{matrix} x_1 = x_3 \\ x_2 = -\frac{3}{2}x_3 \\ x_3 = x_3 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$

$\lambda = 1+2i \quad \begin{bmatrix} 1-(1+2i) & 0 & 0 \\ 2 & 1-(1+2i) & -2 \\ 3 & 2 & 1-(1+2i) \end{bmatrix} = \begin{bmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2i & -2 \\ 0 & 2 & -2i \end{bmatrix}$ $\quad x_1 = 0$

$-\frac{1}{2i}R_1 \to R_1 \qquad -2R_1 + R_2 \Rightarrow R_2 \qquad -3R_1 + R_3 \Rightarrow R_3 \qquad 2x_2 - 2ix_3 = 0$

$\Rightarrow \begin{matrix} x_1 = 0 \\ x_2 = ix_3 \\ x_3 = x_3 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} e^{(1+2i)t} = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} e^t (\cos 2t + i \sin 2t) = e^t \begin{bmatrix} 0 \\ i\cos 2t - \sin 2t \\ \cos 2t + i \sin 2t \end{bmatrix}$

$\vec{x} = c_1 \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{bmatrix} e^t + c_3 \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix} e^t$

Origin repells

12. $\vec{x}' = \begin{bmatrix} -8 & -12 & -6 \\ 2 & 1 & 2 \\ 7 & 12 & 5 \end{bmatrix} \vec{x}$ $\quad \begin{vmatrix} -8-\lambda & -12 & -6 \\ 2 & 1-\lambda & 2 \\ 7 & 12 & 5-\lambda \end{vmatrix} = (-8-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 12 & 5-\lambda \end{vmatrix} + 12 \begin{vmatrix} 2 & 2 \\ 7 & 5-\lambda \end{vmatrix} - 6 \begin{vmatrix} 2 & 1-\lambda \\ 7 & 12 \end{vmatrix}$

$(-8-\lambda)\left[(1-\lambda)(5-\lambda) - 24\right] + 12\left[2(5-\lambda) - 14\right] - 6\left[24 - 7(1-\lambda)\right] =$

$(-8-\lambda)\left[\lambda^2 - 6\lambda + 5 - 24\right] + 12\left[10 - 2\lambda - 14\right] - 6\left[24 - 7 + 7\lambda\right] =$

$(-8-\lambda)\left[\lambda^2 - 6\lambda - 19\right] + 12\left[-2\lambda - 4\right] - 6\left[17 + 7\lambda\right] =$

$-8\lambda^2 - \lambda^3 + 48\lambda + 6\lambda^2 + 152 + 19\lambda - 24\lambda - 48 - 102 - 42\lambda = -\lambda^3 - 2\lambda^2 + \quad \lambda + 2 = 0$

$\lambda^3 + 2\lambda^2 - \quad \lambda - 2 = 0$

$\lambda^2(\lambda+2) - 1(\lambda+2) = 0 \Rightarrow (\lambda+2)(\lambda^2-1) = 0 \Rightarrow (\lambda+2)(\lambda-1)(\lambda+1) = 0 \quad \lambda = -2, 1, -1$

$\lambda_1 = -2 \quad \begin{bmatrix} -8-(-2) & -12 & -6 \\ 2 & 1-(-2) & 2 \\ 7 & 12 & 5-(-2) \end{bmatrix} = \begin{bmatrix} -6 & -12 & -6 \\ 2 & 3 & 2 \\ 7 & 12 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \begin{matrix} x_1 + x_3 = 0 \\ x_2 = 0 \end{matrix} \Rightarrow \quad \begin{matrix} x_1 = -x_3 \\ x_2 = 0 \\ x_3 = x_3 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 1 \quad \begin{bmatrix} -8-1 & -12 & -6 \\ 2 & 1-1 & 2 \\ 7 & 12 & 5-1 \end{bmatrix} = \begin{bmatrix} -9 & -12 & -6 \\ 2 & 0 & 2 \\ 7 & 12 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \begin{matrix} x_1 + x_3 = 0 \\ x_2 - 1/4 x_3 = 0 \end{matrix} \Rightarrow \quad \begin{matrix} x_1 = -x_3 \\ x_2 = 1/4 x_3 \\ x_3 = x_3 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix}$

$\lambda_3 = -1 \quad \begin{bmatrix} -8-(-1) & -12 & -6 \\ 2 & 1-(-1) & 2 \\ 7 & 12 & 5-(-1) \end{bmatrix} = \begin{bmatrix} -7 & -12 & -6 \\ 2 & 2 & 2 \\ 7 & 12 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 6/5 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \begin{matrix} x_1 + 6/5 x_3 = 0 \\ x_2 - 1/5 x_3 = 0 \end{matrix} \Rightarrow \quad \begin{matrix} x_1 = -\frac{6}{5}x_3 \\ x_2 = 1/5 x_3 \\ x_2 = x_3 \end{matrix} \quad \vec{v}_3 = \begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix}$

$\vec{x} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix} e^{-t}$

Saddle point