Homogeneous Solutions & Nullspaces Key

1. a. $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 - x_3 = 7$
$x_2 - x_3 = -1$
$x_3 = \text{free}$

$\Rightarrow$

$x_1 = x_3 + 7$
$x_2 = x_3 - 1$
$x_3 = x_3$

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$$

b. $\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + x_3 = 0$
$x_2 + x_3 = 0$
x_3 free

$\Rightarrow$

$x_1 = -x_3$
$x_2 = -x_3$
$x_3 = x_3$

$$\vec{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3$$

c. $\left[\begin{array}{ccc|c} 5 & -3 & 2 & 0 \\ -3 & -4 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2/29 & 0 \\ 0 & 1 & -16/29 & 0 \end{array}\right]$

$x_1 + \frac{2}{29} x_3 = 0$
$x_2 - \frac{16}{29} x_3 = 0$
x_3 free

$\Rightarrow$

$x_1 = -\frac{2}{29} x_3$
$x_2 = \frac{16}{29} x_3$
$x_3 = x_3$

$$\vec{x} = \begin{bmatrix} -2/29 \\ 16/29 \\ 1 \end{bmatrix} x_3 \quad \text{or if } x_3 = 29t \quad \vec{x} = \begin{bmatrix} -2 \\ 16 \\ 29 \end{bmatrix} t$$

d. $\left[\begin{array}{ccccc|c} 1 & 2 & -3 & 1 & -1 & 1 \\ 2 & -2 & 5 & -2 & -1 & 6 \\ 1 & 4 & -4 & -1 & 2 & 3 \\ -3 & 5 & -9 & -8 & 5 & -7 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -3/2 & 3/2 \\ 0 & 1 & 0 & 0 & 21/10 & 29/18 \\ 0 & 0 & 1 & 0 & 6/5 & 11/9 \\ 0 & 0 & 0 & 1 & -1/10 & -1/18 \end{array}\right]$

$x_1 - \frac{3}{2} x_5 = \frac{3}{2}$
$x_2 + \frac{21}{10} x_5 = \frac{29}{18}$
$x_3 + \frac{6}{5} x_5 = \frac{11}{9}$
$x_4 - \frac{1}{10} x_5 = -\frac{1}{18}$
$x_5 = \text{free}$

$\Rightarrow$

$x_1 = \frac{3}{2} x_5 + \frac{3}{2}$
$x_2 = -\frac{21}{10} x_5 + \frac{29}{18}$
$x_3 = -\frac{6}{5} x_5 + \frac{11}{9}$
$x_4 = \frac{1}{10} x_5 - \frac{1}{18}$
$x_5 = x_5$

1d. cont'd

$$\vec{x} = \begin{bmatrix} 3/2 \\ -21/10 \\ -6/5 \\ 1/10 \\ 1 \end{bmatrix} x_5 + \begin{bmatrix} 3/2 \\ 29/18 \\ 11/9 \\ -1/18 \\ 0 \end{bmatrix} \quad \text{or if } 10t = x_5 \quad \vec{x} = \begin{bmatrix} 15 \\ -21 \\ -12 \\ 1 \\ 10 \end{bmatrix} t + \begin{bmatrix} 3/2 \\ 29/18 \\ 11/9 \\ -1/18 \\ 0 \end{bmatrix}$$

2. a. $\begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & -7 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

$$\begin{aligned} x_1 - 2x_3 - 7x_4 &= 0 \\ x_2 + 2x_3 - 4x_4 &= 0 \\ x_3 &\text{ free} \\ x_4 &\text{ free} \end{aligned} \Rightarrow \begin{aligned} x_1 &= 2x_3 + 7x_4 \\ x_2 &= -2x_3 + 4x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix} x_4 \qquad \text{Nul } A = \text{span}\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

b. $\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{aligned} x_1 - 4x_2 + 5x_6 &= 0 \\ x_3 - x_6 &= 0 \\ x_5 - 4x_6 &= 0 \\ x_2 &\text{ free} \\ x_4 &\text{ free} \\ x_6 &\text{ free} \end{aligned} \Rightarrow \begin{aligned} x_1 &= 4x_2 - 5x_6 \\ x_2 &= x_2 \\ x_3 &= x_6 \\ x_4 &= x_4 \\ x_5 &= 4x_6 \\ x_6 &= x_6 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix} x_6$$

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

2c. $\begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\text{Nul}A = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

$x_1 - x_4 = 0$
$x_2 - x_4 = 0$
$x_3 = 0$
x_4 free

$\Rightarrow$

$x_1 = x_4$
$x_2 = x_4$
$x_3 = 0$
$x_4 = x_4$

$\Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} x_4$

d. $\begin{bmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ 5 & 7 & 2 \end{bmatrix}$ (add a dummy column of zeros for calc) $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = 0$
$x_2 = 0$
$x_3 = 0$

$\Rightarrow \text{Nul}A = \vec{0}$

3. a. $\begin{bmatrix} 2 & 0 & -1 \\ -1 & 3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{6} \end{bmatrix}$

$x_1 - \frac{1}{2}x_3 = 0$
$x_2 + \frac{1}{6}x_3 = 0$
x_3 free

$\Rightarrow$

$x_1 = \frac{1}{2}x_3$
$x_2 = -\frac{1}{6}x_3$
$x_3 = x_3$

$\vec{x} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{6} \\ 1 \end{bmatrix} x_3$ or if $x_3 = 6t$

$\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} t = \text{Ker}T$

b. $\begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\text{Ker}T = \vec{0}$

c. $A = \begin{bmatrix} 1 & 2 & -4 & 3 & -2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 7 \\ 0 & 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -4 & 0 & 0 & 23 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$x_1 + 2x_2 - 4x_3 + 23x_6 = 0$
$x_4 - 3x_6 = 0$
$x_5 + 4x_6 = 0$
$x_7 = 0$

x_2, x_3, x_6 free

$\Rightarrow$

$x_1 = -2x_2 + 4x_3 - 23x_6$
$x_2 = x_2$
$x_3 = x_3$
$x_4 = 3x_6$
$x_5 = -4x_6$
$x_6 = x_6$
$x_7 = 0$

3c contd

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -2 \\ 3 \\ 0 \\ 3 \\ -4 \\ 1 \\ 0 \end{bmatrix} x_6$$

$$\text{Ker } A = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \\ 3 \\ -4 \\ 1 \\ 0 \end{bmatrix} \right\}$$

d. $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - x_3 = 0$
$x_2 = 0$
x_3 free

$\Rightarrow$

$x_1 = x_3$
$x_2 = 0$
$x_3 = x_3$

$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3$

$$\text{Ker } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$