202 Homework #7 key

C

1. to show orthogonal, find dot products. orthogonal $\Rightarrow$ independent.

$\vec{u}_1\cdot\vec{u}_2 = 6-6+0=0 \qquad \vec{u}_2\cdot\vec{u}_3 = 2+2-4=0 \qquad \vec{u}_1\cdot\vec{u}_3 = 3-3+0=0$ ✓

these do form a basis for $\mathbb{R}^3$.

$\|\vec{u}_1\| = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

$\|\vec{u}_2\| = \sqrt{4+4+1} = 3$

$\|\vec{u}_3\| = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$

$$\hat{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \quad \hat{u}_2 = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix} \quad \hat{u}_3 = \begin{bmatrix} 1/3\sqrt{2} \\ 1/3\sqrt{2} \\ 4/3\sqrt{2} \end{bmatrix}$$

$\vec{u}_i\cdot\vec{u}_i = 1$ in orthonormal basis

$\vec{x}\cdot\hat{u}_1 = 5\cdot 1/\sqrt{2} - 3\cdot 1/\sqrt{2} + 0 = 2/\sqrt{2}$

$\vec{x}\cdot\hat{u}_2 = 5\cdot 2/3 - 3(2/3) + 1(-1/3) = \frac{10}{3} - \frac{6}{3} - \frac{1}{3} = \frac{3}{3} = 1$

$\vec{x}\cdot\hat{u}_3 = 5(1/3\sqrt{2}) - 3(1/3\sqrt{2}) + 1(4/3\sqrt{2}) = \frac{5}{3\sqrt{2}} - \frac{3}{3\sqrt{2}} + \frac{4}{3\sqrt{2}} = \frac{6}{3\sqrt{2}} = \frac{2}{\sqrt{2}}$

$$[\vec{x}]_B = \begin{bmatrix} 2/\sqrt{2} \\ 1 \\ 2/\sqrt{2} \end{bmatrix}$$

2. a. $\vec{u}_1\cdot\vec{u}_2 = -2+2-1+1=0 \qquad \vec{u}_1\cdot\vec{u}_3 = 1+2-2-1=0 \qquad \vec{u}_1\cdot\vec{u}_4 = -1+2+1-2=0$

$\vec{u}_3\cdot\vec{u}_4 = -1+1-2+2=0$

$\vec{u}_2\cdot\vec{u}_3 = -2+1+2-1=0 \qquad \vec{u}_2\cdot\vec{u}_4 = 2+1-1-2=0$

$$\frac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \frac{4+10-3+3}{1+4+1+1} \Rightarrow \frac{14}{7}\begin{bmatrix}1\\2\\1\\1\end{bmatrix} = \begin{bmatrix}2\\4\\2\\2\end{bmatrix}$$

$$\frac{\vec{x}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \frac{-8+5+3+3}{4+1+1+1} \Rightarrow \frac{3}{7}\begin{bmatrix}-2\\1\\-1\\1\end{bmatrix} = \begin{bmatrix}-6/7\\3/7\\-3/7\\3/7\end{bmatrix}$$

$$\frac{\vec{x}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} = \frac{4+5+6-3}{1+1+4+1} \Rightarrow \frac{12}{7}\begin{bmatrix}1\\1\\-2\\-1\end{bmatrix} = \begin{bmatrix}12/7\\12/7\\-24/7\\-12/7\end{bmatrix}$$

$$\vec{y}_{\parallel} = \begin{bmatrix}2\\4\\2\\2\end{bmatrix} + \begin{bmatrix}-6/7\\3/7\\-3/7\\3/7\end{bmatrix} + \begin{bmatrix}12/7\\12/7\\-24/7\\-12/7\end{bmatrix} = \begin{bmatrix}20/7\\43/7\\-13/7\\5/7\end{bmatrix}$$

$$\vec{y}_{\perp} = \begin{bmatrix}4\\5\\-3\\3\end{bmatrix} - \begin{bmatrix}20/7\\43/7\\-13/7\\5/7\end{bmatrix} = \begin{bmatrix}8/7\\-8/7\\-8/7\\16/7\end{bmatrix}$$

which is a multiple of $\vec{u}_4$ in $W^{\perp}$

$\vec{y}_{\parallel}$ best approx.

distance $= \|\vec{y}_{\perp}\| = \sqrt{(8/7)^2+(8/7)^2+(8/7)^2+(16/7)^2} = \sqrt{\frac{64}{7}} = \frac{8}{\sqrt{7}}$

b. $\vec{u}_1\cdot\vec{u}_2 = -1+1+0=0$

$$\frac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \frac{-1+4+0}{1+1} \Rightarrow \frac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}3/2\\3/2\\0\end{bmatrix}$$

$$\frac{\vec{x}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \frac{1+4+0}{1+1} \Rightarrow \frac{5}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}-5/2\\5/2\\0\end{bmatrix}$$

$$\vec{y}_{\parallel} = \begin{bmatrix}3/2\\3/2\\0\end{bmatrix} + \begin{bmatrix}-5/2\\5/2\\0\end{bmatrix} = \begin{bmatrix}-1\\4\\0\end{bmatrix} \qquad \vec{y}_{\perp} = \begin{bmatrix}-1\\4\\3\end{bmatrix} - \begin{bmatrix}-1\\4\\0\end{bmatrix} = \begin{bmatrix}0\\0\\3\end{bmatrix} \text{ in } W^{\perp}$$

best approx in W

$\|\vec{y}_{\perp}\| = \sqrt{3^2} = 3$

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2c. $\vec{u_1}\cdot\vec{u_2} = 1+0+0-1=0$ $\quad \vec{u_1}\cdot\vec{u_3} = 0-1+0+1=0$ $\quad \vec{u_2}\cdot\vec{u_3} = 0+0+1-1=0$

$$\frac{\vec{x}\cdot\vec{u_1}}{\vec{u_1}\cdot\vec{u_1}} = \frac{3+4+0-6}{1+1+1} \Rightarrow \frac{1}{3}\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} = \begin{bmatrix}1/3\\1/3\\0\\-1/3\end{bmatrix}$$

$$\frac{\vec{x}\cdot\vec{u_2}}{\vec{u_2}\cdot\vec{u_2}} = \frac{3+0+5+6}{1+1+1} \Rightarrow \frac{14}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}14/3\\0\\14/3\\14/3\end{bmatrix}$$

$$\frac{x\cdot\vec{u_3}}{\vec{u_3}\cdot\vec{u_3}} = \frac{0-4+5-6}{1+1+1} \Rightarrow \frac{-5}{3}\begin{bmatrix}0\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\5/3\\-5/3\\5/3\end{bmatrix}$$

$$\vec{y}_{\parallel} = \begin{bmatrix}1/3\\1/3\\0\\-1/3\end{bmatrix} + \begin{bmatrix}14/3\\0\\14/3\\14/3\end{bmatrix} + \begin{bmatrix}0\\5/3\\-5/3\\5/3\end{bmatrix} = \begin{bmatrix}5\\2\\3\\6\end{bmatrix}$$

$$\vec{y}_{\perp} = \begin{bmatrix}3\\4\\5\\6\end{bmatrix} - \begin{bmatrix}5\\2\\3\\6\end{bmatrix} = \begin{bmatrix}-2\\2\\2\\0\end{bmatrix}$$

3. a. $\vec{v_1} = \begin{bmatrix}2\\-5\\1\end{bmatrix}$ $\quad \dfrac{\vec{v_1}\cdot\vec{u_2}}{\vec{v_1}\cdot\vec{v_1}}\underbrace{\begin{bmatrix}2\\-5\\1\end{bmatrix}}_{\vec{v_1}} = \dfrac{6+5+2}{4+25+1}\begin{bmatrix}2\\-5\\1\end{bmatrix} = \dfrac{12}{30}\begin{bmatrix}2\\-5\\1\end{bmatrix} = \begin{bmatrix}/30\\/30\\/30\end{bmatrix} = \begin{bmatrix}4/5\\-2\\2/5\end{bmatrix}$

$$\vec{v_2} = \begin{bmatrix}4\\-1\\2\end{bmatrix} - \begin{bmatrix}4/5\\-2\\2/5\end{bmatrix} = \begin{bmatrix}16/5\\1\\8/5\end{bmatrix} \quad \left\{\begin{bmatrix}2\\-5\\1\end{bmatrix}, \begin{bmatrix}16\\5\\8\end{bmatrix}\right\}$$

b. $\vec{v_1} = \begin{bmatrix}3\\-1\\2\\-1\end{bmatrix}$ $\quad \dfrac{\vec{v_1}\cdot\vec{u_2}}{\vec{v_1}\cdot\vec{v_1}}\underbrace{\begin{bmatrix}3\\-1\\2\\-1\end{bmatrix}}_{\vec{v_1}} = \dfrac{-15-9-18-3}{9+1+4+1}\begin{bmatrix}3\\-1\\2\\-1\end{bmatrix} = \dfrac{-45}{15}\begin{bmatrix}3\\-1\\2\\-1\end{bmatrix} = \begin{bmatrix}-9\\3\\-6\\3\end{bmatrix}$

$$\vec{v_2} = \begin{bmatrix}-5\\9\\-9\\3\end{bmatrix} - \begin{bmatrix}-9\\3\\-6\\3\end{bmatrix} = \begin{bmatrix}4\\6\\-3\\0\end{bmatrix} \qquad \left\{\begin{bmatrix}3\\-1\\2\\-1\end{bmatrix}, \begin{bmatrix}4\\6\\-3\\0\end{bmatrix}\right\}$$

4. $A = QR \Rightarrow Q^TA = Q^TQR = R$

$$\begin{bmatrix}5/6 & 1/6 & -3/6 & 1/6\\ -1/6 & 5/6 & 1/6 & 3/6\end{bmatrix}\begin{bmatrix}5 & 9\\1 & 7\\-3 & -5\\1 & 5\end{bmatrix} = \begin{bmatrix}\frac{25}{6}+\frac{1}{6}+\frac{9}{6}+\frac{1}{6} & \frac{45}{6}+\frac{7}{6}+\frac{15}{6}+\frac{5}{6}\\ -\frac{5}{6}+\frac{5}{6}-\frac{3}{6}+\frac{3}{6} & -\frac{9}{6}+\frac{35}{6}-\frac{5}{6}+\frac{15}{6}\end{bmatrix} = \begin{bmatrix}6 & 12\\0 & 6\end{bmatrix} = R$$

5. a. false U^TU is .
b. true
c. true
d. true
e. true
f. false it's $\mathbb{R}^5$; the range is $\mathbb{R}^3$.
g. true (if we allow for infinite dimensional matrices)

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5h. true

i. true

j. false; they are always linear.

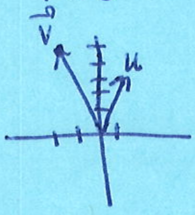
k. false; it can be one-to-one, but cannot be onto.

l. true

6a. $\left[\begin{array}{ccc|c} 1 & -2 & 3 & -6 \\ 2 & -5 & 6 & -4 \end{array}\right] \Rightarrow \text{rref} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & -22 \\ 0 & 1 & 0 & -8 \end{array}\right]$ so x is not unique matrix is dependent.

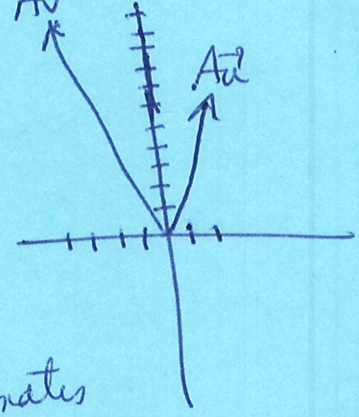
b. $\left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 3 & -8 & 8 & 6 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 8 & 10 \end{array}\right]$ rref $\Rightarrow \begin{bmatrix} 1 & 0 & 8 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so x is not unique

7. 7×5

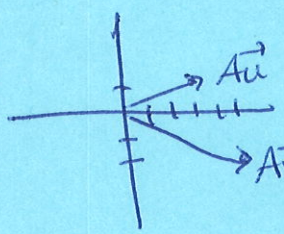
8. 

a. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $\quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ 10 \end{bmatrix}$

doubles length of vector

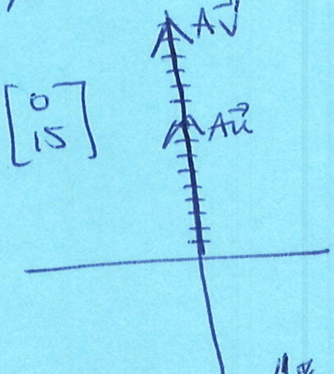


b. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ swaps coordinates reflects over $y=x$



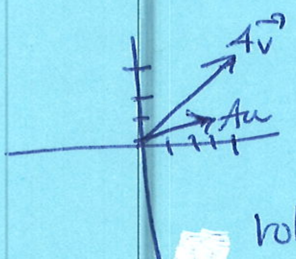
c. $\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \end{bmatrix}$

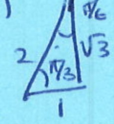
projects onto y-axis & stretches



d. $\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 + \frac{3\sqrt{3}}{2} \\ -\sqrt{3}/2 + 3/2 \end{bmatrix} \approx \begin{bmatrix} 3.1 \\ .63 \end{bmatrix}$

$\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -2/2 + 5\sqrt{3}/2 \\ 2\sqrt{3}/2 + 5/2 \end{bmatrix} \approx \begin{bmatrix} 4.4 \\ 4.2 \end{bmatrix}$

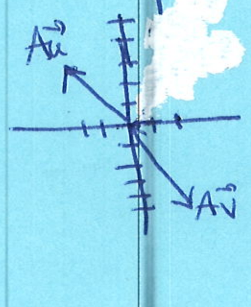




rotates clockwise 60°

e. $\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} - 3/\sqrt{2} \\ 1/\sqrt{2} - 3/\sqrt{2} \end{bmatrix} \approx \begin{bmatrix} -2.3 \\ 2.8 \end{bmatrix}$

$\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} - 5/\sqrt{2} \\ -2/\sqrt{2} - 5/\sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 2.1 \\ -4.9 \end{bmatrix}$



rotates clockwise 225°

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9. $T(\vec{x}) = A\vec{x} + \vec{b}$. This kind of transformation is not linear whenever $\vec{b} \neq 0$.

$T(0) = A(0) + \vec{b} = \vec{b}$ does not map $\vec{0}$ to $\vec{0}$

$T(\vec{x}) + T(\vec{y}) = A\vec{x} + A\vec{y} + 2\vec{b}$ but $T(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} + \vec{b}$

These are not equal.

$T(k\vec{x}) = Ak\vec{x} + \vec{b} = kA\vec{x} + \vec{b}$ but $kT(\vec{x}) = k(A\vec{x} + \vec{b}) = kA\vec{x} + k\vec{b}$