202 Proof Set #4

1. Consider the eigenvalue equation $A\vec{x} = \lambda\vec{x}$. for $A = P$ and $\vec{x} = \vec{q}$. it seems pretty clear that $\lambda = 1$ in this relationship. Then by definition of eigenvalues and eigenvectors, $\lambda = 1$ is the eigenvalue and $\vec{q}$ is an eigenvector for P. //

2. Consider $A\vec{x} = \lambda\vec{x}$. if we take the transpose of both sides we get $(A\vec{x})^T = (\lambda\vec{x})^T \Rightarrow \vec{x}^T A^T = \vec{x}^T \cdot \lambda$. Though the transpose multiplies the vector on the left, λ is still the eigenvalue of A^T and $\vec{x}^T$ is an eigenvector. //

3. Consider $A\vec{x} = \lambda\vec{x}$. if we multiply both sides of the equation by A^{-1} we get $(A^{-1}A)\vec{x} = A^{-1}(\lambda\vec{x})$ or $I\vec{x} = \lambda(A^{-1}\vec{x})$. since we can divide by λ if $\lambda \neq 0$ (which is required if A is invertible) we get $A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$. This $1/\lambda$ is an eigenvalue of A^{-1}. //

4. If A and B are similar, then if $A\vec{x} = \lambda x$ and $A = PBP^{-1}$, we have $(PBP^{-1})\vec{x} = \lambda\vec{x}$. Multiplying both sides by P^{-1} we get $P^{-1}PBP^{-1}\vec{x} = BP^{-1}\vec{x} = P^{-1}(\lambda\vec{x}) = \lambda P^{-1}\vec{x}$. or $B(P^{-1}\vec{x}) = \lambda(P^{-1}\vec{x})$ thus $P^{-1}\vec{x}$ is the eigenvector of B, but λ is the corresponding eigenvalue of B, which is the same eigenvalue as A. //

5. To show that $A = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}$ is diagonalizable (in real numbers), then A must have distinct eigenvalues, or if the eigenvalue is repeated, it must have a 2-dimensional eigenspace. The characteristic polynomial of A is $(1-\lambda)(-6-\lambda) + 12 = \lambda^2 + 5\lambda - 6 + 12 = \lambda^2 + 5\lambda + 6 = 0$ which implies that $(\lambda+2)(\lambda+3) = 0$ or that $\lambda = -2$, and $\lambda = -3$. Since the eigenvalues are distinct, A is diagonalizable. //

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6. Suppose that A is both diagonalizable and invertible. If A is invertible no $\lambda = 0$, and if A is diagonalizable then $A = PDP^{-1}$. If we take the inverse of both sides we find that $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1}$ which implies $A^{-1} = PD^{-1}P^{-1}$. From an earlier proof we know that if λ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A with the same eigenvectors. If $A^{-1} = PD^{-1}P^{-1}$, then D^{-1} is a diagonal matrix w/ $1/\lambda_i$ on the diagonal, and no $1/\lambda$ can be equal zero. Thus $\det(A^{-1}) \neq 0$ so A^{-1} is invertible, and D^{-1} is the diagonal matrix of A^{-1}, so A^{-1} is diagonalizable. //

7. Consider the matrix $D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$ w/ $P = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$. We can construct an $A = PDP^{-1} = \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix}$ so A is similar to D, which is diagonal so A is diagonalizable, but A is not invertible since D is not invertible, and since $\det A = 0$. //

8. If λ is an eigenvalue of A then $A\vec{x} = \lambda\vec{x}$. If we add $-c\vec{x}$ to both sides of the equation we get $A\vec{x} - c\vec{x} = \lambda\vec{x} - c\vec{x}$ or $(A - cI)\vec{x} = (\lambda - c)\vec{x}$. By definition of an eigenvalue, the matrix $(A - cI)$ has an eigenvalue of $\lambda - c$. //

9. Suppose that A is diagonalizable with real eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$. Then there exists a P such that $A = PDP^{-1}$ with λ_i's on the diagonal. If we take the determinant of both sides we get $\det A = \det(PDP^{-1}) = \det P \cdot \det D \cdot \det P^{-1}$ by properties of determinants, which in turn equals $\det A = \det P \cdot \det P^{-1} \det D = \det(PP^{-1}) \det P = \det I \cdot \det D = 1 \cdot \det D$. Thus $\det A = \det D$, but since D is diagonal, $\det D$ is the product of its diagonal entries or $\prod_{i=1}^{n} \lambda_i$, thus $\det A = \prod_{i=1}^{n} \lambda_i$. //

10. Suppose that $A\vec{x} = \lambda\vec{x}$. If we multiply this expression by A on both sides we get $A(A\vec{x}) = A(\lambda\vec{x})$, which is the same as $A^2\vec{x} = \lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}$. Suppose λ^n is an eigenvalue of A^n. Then for A^{n+1} we have $A^{n+1}\vec{x} = A(A^n)\vec{x} = A(\lambda^n\vec{x}) = \lambda^n(A\vec{x}) = \lambda^n(\lambda\vec{x})$ $= \lambda^{n+1}\vec{x}$, thus λ^{n+1} is an eigenvalue of A^n if λ^n is an eigenvalue of A^n. Thus for every integer $n \geq 2$, if λ is an eigenvalue of A, then λ^n is an eigenvalue of A^n. //

11. Suppose that $\det A = 0$. Since $\det A$ is the product of the eigenvalues of A, then at least one eigenvalue of A must be zero. If one (or more) eigenvalues of A is zero, then the matrix is not invertible since A cannot be row-equivalent to the identity, so A is singular. If A is singular, then by definition, the determinant is zero. //

12. Suppose that the eigenvalues of a diagonalizable matrix are all ± 1. Then $A = PDP^{-1}$ and so $A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}$, so A^{-1} diagonalizes to D^{-1} w/ $1/\lambda_i$ on the diagonal. But if all λ's are ± 1 then $1/\lambda_i = \lambda_i$ for all i. Thus $D^{-1} = D$. But if $D^{-1} = D$ then $A^{-1} = PDP^{-1} = A$. So $A^{-1} = A$. //