MAT 212, Final Exam, Summer 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Estimate the solution of the ODE $y' = x^{2/3} + y^{2/3}$ using $\Delta t = 0.2$ with the indicated method and number of steps.

$y(0) = 1$

a. Euler's method, 3 steps (8 points)

n	x_n	y_n	f_n	$y_n + .2(f_n)$
0	0	1	$0^{2/3} + 1^{2/3} = 1$	$1 + .2(1) = 1.2$
1	.2	1.2	$.2^{2/3} + 1.2^{2/3} = 1.4712...$	$1.2 + .2(1.4712...) = 1.4942...$
2	.4	1.494	$.4^{2/3} + 1.494^{2/3} = 1.8499$	$1.4942 + .2(1.8499) = 1.8412...$
3	.6	1.84...		

$y_3 = 1.8412$

b. Improved Euler's, 3 steps (8 points)

n	x_n	y_n	k_1	u_{n+1}	k_2	$y_{n+1} = y_n + \frac{1}{2}(.2)(k_1 + k_2)$
0	0	1	1	1.2	$.2^{2/3} + 1.2^{2/3} = 1.4712$	$1 + .1(1 + 1.47) = 1.247...$
1	.2	1.247	1.5006	1.547246	1.8806...	1.585123905
2	.4	1.585...	1.90237	1.965598...	2.2805...	2.0034...
3	.6	2.0034				

$Y_3 = 2.0034$

c. Runge-Kutta, 1 step (8 points)

$x_0 = 0, \; y_0 = 1$

$k_{01} = 0^{2/3} + 1^{2/3} = 1 \quad t_0 + .1 = .1 \quad y_0 + \frac{1}{2}(.2)k_{01} = 1 + .1(1) = 1.1$

$k_{02} = .1^{2/3} + 1.1^{2/3} = 1.281045706 \quad y_0 + \frac{1}{2}(.2)k_{02} = 1 + .1(1.281...) = 1.128104571...$

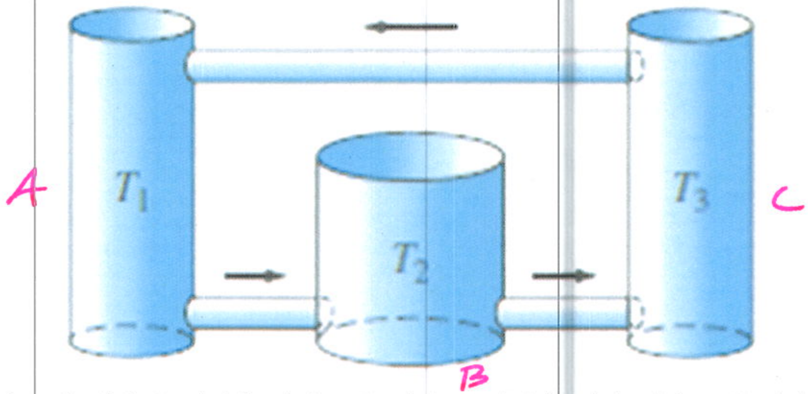
$k_{03} = .1^{2/3} + 1.128...^{2/3} = 1.299119762 \quad y_0 + .2(1.299119762) = 1.259823952$

$k_{n4} = .2^{2/3} + 1.2598^{2/3} = 1.508464295$

$y_1 = 1 + .2(\frac{1}{6})(1 + 1.28... * 2 + 2 * 1.299... + 1.50846) = 1.255626508$

$y_1 \approx 1.2556$

2. Three 100 gallon fermentation vats are connected as shown.



(Tank 1 feeds into Tank 2; Tank 2 feeds into Tank 3; Tank 3 feeds back into Tank 1.)
If the mixture is well-stirred and circulates between the tanks at the rate of 10 gal/min, derive the set of equations to model the system. (You don't need to solve.) (8 points)

$$\frac{dA}{dt} = -\frac{A}{100\,\text{gal}} \cdot \frac{10\,\text{gal}}{\text{min}} + \frac{C}{100}\frac{10}{\text{min}} \Rightarrow A' = -\frac{1}{10}A + \frac{1}{10}C$$

$$\frac{dB}{dt} = \frac{A \cdot 10}{100} - \frac{B}{100} \cdot \frac{10}{\text{min}} \Rightarrow B' = \frac{1}{10}A - \frac{1}{10}B$$

$$\frac{dC}{dt} = \frac{B \cdot 10}{100} - \frac{C \cdot 10}{100} \Rightarrow C' = \frac{1}{10}B - \frac{1}{10}C$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix}' = \begin{bmatrix} -1/10 & 0 & 1/10 \\ 1/10 & -1/10 & 0 \\ 0 & 1/10 & -1/10 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

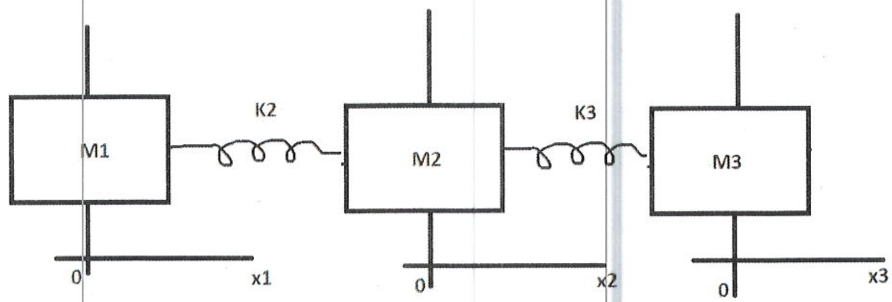
3. Rewrite $y''' + 2y'' + y' + y = 1 + te^t$ as a system of first order equations. (You don't need to solve.) (8 points)

$y = x_1$
$y' = x_2 = x_1'$
$y'' = x_3 = x_2'$
$y''' = x_3'$
$x_3' + 2x_3 + x_2 + x_1 = 1 + te^t$

$x_1' = x_2$
$x_2' = x_3$
$x_3' = -x_1 - x_2 - 2x_3 + 1 + te^t$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 1+te^t \end{bmatrix}$$

4. Set up the spring problem shown below as a system of equations. (You don't need to solve.) (10 points)



$$m_1 x_1'' = -k_2 x_1 + k_2 x_2$$

$$m_2 x_2'' = -k_2 x_2 - k_3 x_2 + k_2 x_1 + k_3 x_3$$

$$m_3 x_3'' = -k_3 x_3 + k_2 x_2$$

$$\vec{x}'' = \begin{bmatrix} -\frac{k_2}{m_1} & \frac{k_2}{m_1} & 0 \\ \frac{k_2}{m_2} & \frac{-k_2-k_3}{m_2} & \frac{k_3}{m_2} \\ 0 & \frac{k_2}{m_3} & -\frac{k_3}{m_3} \end{bmatrix} \vec{x}$$

5. Solve $\vec{x}' = \begin{pmatrix} 1 & 9 \\ -2 & -5 \end{pmatrix} \vec{x}$. Write the general solution (with real terms only). Plot several sample trajectories. (10 points)

$$(1-\lambda)(-5-\lambda) + 18 = 0$$

$$\lambda^2 + 4\lambda - 5 + 18 = 0$$

$$\lambda^2 + 4\lambda + 13 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

$$\begin{bmatrix} 1+2-3i & 9 \\ -2 & -5+2-3i \end{bmatrix} = \begin{bmatrix} 3-3i & 9 \\ -2 & -3-3i \end{bmatrix}$$

$$\frac{-2x_1}{-2} = \frac{(3+3i)x_2}{-2}$$

$$x_1 = \frac{(+3+3i)}{-2} x_2 \qquad \vec{v}_1 = \begin{pmatrix} 3+3i \\ -2 \end{pmatrix}$$

$$x_2 = x_2$$

$$e^{-2t}\begin{pmatrix} 3+3i \\ -2 \end{pmatrix}(\cos 3t + i \sin 3t) = e^{-2t}\begin{pmatrix} 3\cos 3t + 3i\sin 3t + 3i\cos 3t - 3\sin 3t \\ -2\cos 3t - 2i\sin 3t \end{pmatrix}$$

$$\vec{x}(t) = c_1 e^{-2t}\begin{pmatrix} 3\cos 3t - 3\sin 3t \\ -2\cos 3t \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 3\sin 3t + 3\cos 3t \\ -2\sin 3t \end{pmatrix}$$

6. Find the general solution to the system $t\vec{x}' = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\vec{x}$. Write the fundamental solution matrix. (8 points)

$$(-1-\lambda)(-1-\lambda)+2=0$$
$$\lambda^2+2\lambda+1+2=0$$
$$\lambda^2+2\lambda+3=0$$

$$\lambda = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2\pm 2\sqrt{2}i}{2}$$

$$\lambda = -1 \pm \sqrt{2}i$$

$$t^{-1+\sqrt{2}i} = t^{-1}t^{\sqrt{2}i} = t^{-1}e^{\ln t \cdot \sqrt{2}i}$$

$$\begin{bmatrix} \cancel{-1+1}-\sqrt{2}i & -1 \\ 2 & \cancel{-1+1}-\sqrt{2}i \end{bmatrix}$$

$$2x_1 = \sqrt{2}i\,x_2 \qquad \vec{v}_1 = \begin{pmatrix} \sqrt{2}i \\ 2 \end{pmatrix}$$
$$x_1 = \frac{\sqrt{2}}{2}i\,x_2$$
$$x_2 = x_2$$

$$t^{-1}\begin{pmatrix} \sqrt{2}i \\ 2 \end{pmatrix}\left(\cos(\sqrt{2}\ln t) + i\sin(\sqrt{2}\ln t)\right)$$

$$\frac{1}{t}\begin{pmatrix} \sqrt{2}i\cos(\sqrt{2}\ln t) - \sqrt{2}\sin(\sqrt{2}\ln t) \\ 2\cos(\sqrt{2}\ln t) + 2i\sin(\sqrt{2}\ln t) \end{pmatrix}$$

$$\boxed{\vec{X}(t) = \frac{c_1}{t}\begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}\ln t) \\ 2\cos(\sqrt{2}\ln t) \end{pmatrix} + \frac{c_2}{t}\begin{pmatrix} \sqrt{2}\cos(\sqrt{2}\ln t) \\ 2\sin(\sqrt{2}\ln t) \end{pmatrix}}$$

7. Verify that $\Psi = \begin{pmatrix} 3e^{-2t} & e^t & e^{3t} \\ -2e^{-2t} & -e^t & -e^{3t} \\ 2e^{-2t} & e^t & 0 \end{pmatrix}$ is a solution to $\vec{x}' = \begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix}\vec{x}$. Does the solution represent a fundamental set? (8 points)

$$\Psi' = \begin{pmatrix} -6e^{-2t} & e^t & 3e^{3t} \\ 4e^{-2t} & -e^t & -3e^{3t} \\ -4e^{-2t} & e^t & 0 \end{pmatrix}$$

$$W = e^{3t}\begin{vmatrix} -2e^{-2t} & -e^t \\ 2e^{-2t} & e^t \end{vmatrix} + e^{3t}\begin{vmatrix} 3e^{-2t} & e^t \\ 2e^{-2t} & e^t \end{vmatrix} + 0$$
$$= e^{3t}\cancel{(-2e^{-t}+2e^{-t})}^{0} + e^{3t}(3e^{-t}-2e^{-t}) =$$
$$e^{3t}(e^{-t}) = e^{2t}$$

fundamental set

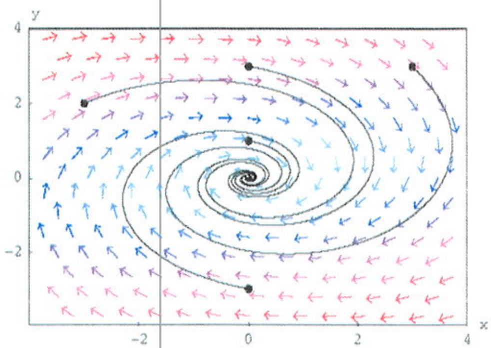
$$\begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix}\begin{pmatrix} 3e^{-2t} & e^t & e^{3t} \\ -2e^{-2t} & -e^t & -e^{3t} \\ 2e^{-2t} & e^t & 0 \end{pmatrix} =$$

$$\begin{pmatrix} -24e^{-2t}+22e^{-2t}-4e^{-2t} & -8e^t+11e^t-2e^t & -8e^{3t}+11e^{3t}+0 \\ 18e^{-2t}-18e^{-2t}+4e^{-2t} & +6e^t-9e^t+2e^t & 6e^{3t}-9e^{3t}+0 \\ -18e^{-2t}+12e^{-2t}+2e^{-2t} & -6e^t+6e^t+e^t & -6e^{3t}+6e^{3t}+0 \end{pmatrix}$$

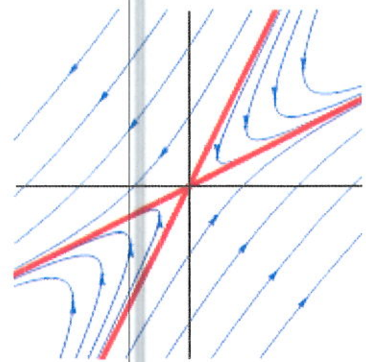
$$\begin{pmatrix} -6e^{-2t} & e^t & 3e^{3t} \\ 4e^{-2t} & -e^t & -3e^{3t} \\ -4e^{-2t} & e^t & 0 \end{pmatrix}$$

Satisfies system

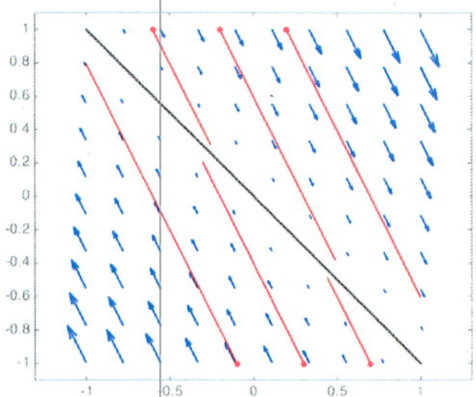
8. For each set of solution curves shown below, match the graphs with proposed solutions and characterize the system as containing a stable vector/orbit, origin attracts, origin repels, origin is a saddle point. (12 points)



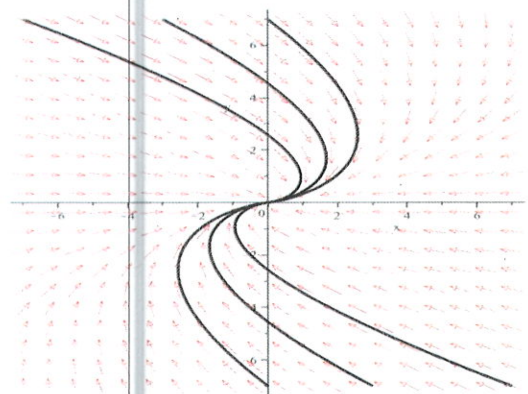
a.



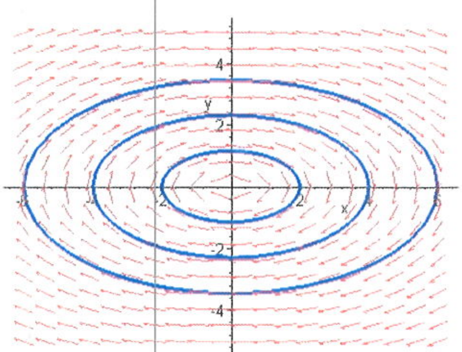
d.



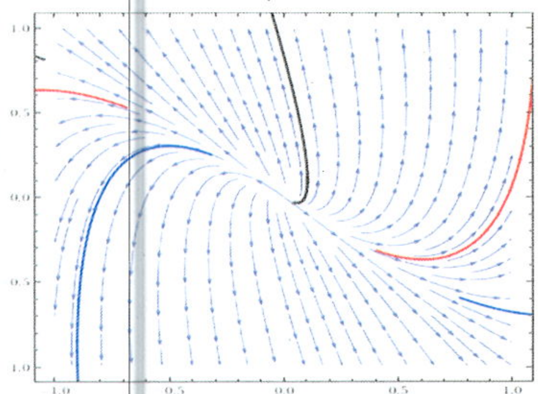
b.



e.



c.



f.

i. $\vec{x}(t) = c_1 \begin{pmatrix} 2\cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ C

ii. $\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} 3\cos t + \sin t \\ 2\sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -3\sin t + \cos t \\ -2\cos t \end{pmatrix}$ A

iii. $\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$ D

iv. $\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-2t}$ B

v. $\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{2t}$ F

vi. $\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t}$ E

9. The fundamental solution matrix to the system $\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$ is $\Psi = \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}$. Use this fact to solve the system $\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 4\cos t \\ 6\sin t \end{pmatrix}$. Recall that $\vec{x}_p = \Psi \int \Psi^{-1} f(t)dt$ or use the method of undetermined coefficients. (10 points)

$W = \cos^2 t - \cancel{2\cos t \sin t} + \cancel{2\cos t \sin t} - 4\sin^2 t + 5\sin^2 t = 1$

$\Psi^{-1} = \frac{1}{1}\begin{pmatrix} \cos t - 2\sin t & 5\sin t \\ -\sin t & \cos t + 2\sin t \end{pmatrix}$ $\quad \Psi^{-1} g = \begin{pmatrix} \cos t - 2\sin t & 5\sin t \\ -\sin t & \cos t + 2\sin t \end{pmatrix}\begin{pmatrix} 4\cos t \\ 6\sin t \end{pmatrix} =$

$= \begin{pmatrix} 4\cos^2 t - 8\sin t\cos t + 30\sin^2 t \\ -4\cos t\sin t + 6\cos t\sin t + 12\sin^2 t \end{pmatrix} = \begin{pmatrix} 4 - 8\sin t\cos t + 26\sin^2 t \\ 12\sin^2 t + 2\cos t\sin t \end{pmatrix} = \begin{pmatrix} 4 - 8\sin t\cos t + 13 - 13\cos 2t \\ 6 - 6\cos 2t + 2\cos t\sin t \end{pmatrix}$

$\int \Psi^{-1} g = \int \begin{pmatrix} 17 - 8\sin t\cos t - 13\cos 2t \\ 6 - 6\cos 2t + 2\sin t\cos t \end{pmatrix} dt = \begin{pmatrix} 17t - 4\sin^2 t - \frac{13}{2}\sin 2t \\ 6t - 3\sin 2t + \sin^2 t \end{pmatrix} = \begin{pmatrix} 17t - 4\sin^2 t - 13\sin t\cos t \\ 6t - 6\sin t\cos t + \sin^2 t \end{pmatrix}$

$\Psi \int \Psi^{-1} g\,dt = \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}\begin{pmatrix} 17t - 4\sin^2 t - 13\sin t\cos t \\ 6t - 6\sin t\cos t + \sin^2 t \end{pmatrix} =$

$\begin{pmatrix} 17t\cos t - 4\cos t\cancel{\sin^2 t} - 13\sin t\cos^2 t + 34t\sin t - 8\sin^3 t - \cancel{26\sin^2 t\cos t} - 30t\sin t + \cancel{30\sin^2 t\cos t} - 5\sin^3 t \\ 17t\sin t - 4\sin^3 t - 13\cancel{\sin^2 t\cos t} + 6t\cos t - 6\sin t\cos^2 t + \cancel{\sin^2 t\cos t} - 12t\sin t + \cancel{12\sin^2 t\cos t} - 2\sin^3 t \end{pmatrix}$

$= \begin{pmatrix} 17t\cos t + 4t\sin t \underbrace{- 13\sin^3 t - 13\sin t\cos^2 t}_{-13\sin t(\sin^2 t + \cos^2 t)} \\ 6t\cos t + 5t\sin t \underbrace{- 6\sin^3 t - 6\sin t\cos^2 t}_{-6\sin t(\sin^2 t + \cos^2 t)} \end{pmatrix} = \begin{pmatrix} 17t\cos t + 4t\sin t - 13\sin t \\ 6t\cos t + 5t\sin t - 6\sin t \end{pmatrix}$

$$\boxed{\vec{X}(t) = \begin{pmatrix} \cos t - 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} 17t\cos t + 4t\sin t - 13\sin t \\ 6t\cos t + 5t\sin t - 6\sin t \end{pmatrix}}$$

10. Use the table of Laplace transforms to find the transforms of the following functions. (3 points each)

a. $t^2 e^t$

$\Rightarrow \frac{2}{(s-1)^3}$

b. $u_3(t) = u(t-3)$ $\Rightarrow \frac{e^{-3s}}{s}$

c. $\int_0^t (t-\tau)^3 \sin\tau \, d\tau$

$f(t) = t^3 \quad g(t) = \sin t$

$$\Rightarrow \frac{6}{s^4} \cdot \frac{1}{s^2+1} = \frac{6}{s^4(s^2+1)}$$

11. Use the table of Laplace transforms to find the inverse Laplace transform of the following function. (3 points each)

a. $\frac{2s-4}{s^2+9} = \frac{2s}{s^2+9} - \frac{4}{s^2+9} = 2\left(\frac{s}{s^2+9}\right) - \frac{4}{3}\left(\frac{3}{s^2+9}\right)$

$$\Rightarrow 2\cos 3t - \frac{4}{3}\sin 3t$$

b. $\frac{1}{s^7} = \frac{1}{6!}\left(\frac{6!}{s^7}\right) \Rightarrow \frac{1}{720} t^6$

c. $\frac{e^{-2s}(s-2)}{(s-2)^2+16} = e^{-2s}\left(\frac{(s-2)}{(s-2)^2+16}\right) \Rightarrow u(t-2)\cos(4(t-2))e^{2t}$

d. $\frac{1}{s^2(s^2-1)} = \frac{1}{s^2} \cdot \frac{1}{s^2-1}$

$$\Rightarrow \int_0^t (t-\tau)\sinh\tau \, d\tau$$

(or use partial fractions)

$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}$

$A + Bt + Ce^t + De^{-t}$

solve for A, B, C, D

12. Write the function $f(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t < \pi \\ \sin t, & t \ge \pi \end{cases}$ in terms of the unit step function. (5 points)

$$t\,u(t) + (\sin t - t)\,u(t-\pi)$$

13. Use Laplace transforms to solve the IVP $y'' + 2y' + 2y = \cos t$, $y(0) = 1$, $y'(0) = 2$. (10 points)

$$s^2Y(s) - s - 2 + 2(sY(s) - 1) + 2Y(s) = \frac{s}{s^2+1}$$

$$Y(s) = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{(s+4)(s^2+1)}{(s^2+2s+2)(s^2+1)} = \frac{s^3+4s^2+2s+4}{(s^2+2s+2)(s^2+1)}$$

$$\frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+1} = \frac{As^3+Bs^2+As+B+Cs^3+2Cs^2+2Cs+Ds^2+2Ds+2D}{(s^2+2s+2)(s^2+1)}$$

$A+C=1$
$B+2C+D=4$
$A+2C+2D=2$
$B+2D=4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 1 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 4 \end{array}\right]$$

$A = 4/5$
$B = 16/5$
$C = 1/5$
$D = 2/5$

$$\frac{4}{5}\left(\frac{s^{+1}}{(s+1)^2+1}\right) + \frac{16}{5}\left(\frac{1}{(s+1)^2+1}\right)^{-1} + \frac{1}{5}\left(\frac{s}{s^2+1}\right) + \frac{2}{5}\left(\frac{1}{s^2+1}\right)$$

$$\frac{4}{5}\left(\frac{s+1}{(s+1)^2+1}\right) + \frac{12}{5}\left(\frac{1}{(s+1)^2+1}\right) + \frac{1}{5}\left(\frac{s}{s^2+1}\right) + \frac{2}{5}\left(\frac{1}{s^2+1}\right) \Rightarrow$$

$$y(t) = \frac{4}{5}e^{-t}\cos t + \frac{12}{5}e^{-t}\sin t + \frac{1}{5}\cos t + \frac{2}{5}\sin t$$

14. For each equation, identify any singular points and classify them as regular or irregular. (5 points each)

a. $3x^2y'' + 2xy' + (1-x)^2y = 0$

$\overline{3x^2}$

$$y'' + \frac{2}{3x}y' + \frac{(1-x)^2}{3x^2}y = 0$$

$x=0$ singular

$\boxed{x=0 \text{ is regular}}$

$\lim_{x\to 0} \frac{2}{3\cancel{x}}\cdot\cancel{x} =$ defined

$\lim_{x\to 0} \frac{(1-x)^2}{3\cancel{x^2}}\cdot\cancel{x^2} =$ defined

b. $x^2y'' + (x - x^3)y' + (\cos x)y = 0$

x^2

$$y'' + \frac{1-x^2}{x}y' + \frac{\cos x}{x^2}y = 0$$

$x=0$ singular

$\boxed{x=0 \text{ is regular}}$

$$\lim_{x\to 0} \frac{1-x^2}{\cancel{x}}\cdot\cancel{x} = \text{defined}$$

$$\lim_{x\to 0} \frac{\cos x}{\cancel{x^2}}\cdot\cancel{x^2} = \text{defined}$$

15. Use series solution methods to find the solution to $(x-2)y'' + 4xy = 0$. State at least 4 terms of each solution (unless it is finite). Be sure that you find ~~three~~ two solutions. (12 points)

$$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$$

$$x\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + 4x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-1} - \sum_{n=2}^{\infty} 2a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} 4a_n x^{n+1} = 0$$

$$\sum_{n=1}^{\infty} a_{n+1}(n+1)n\,x^n - \sum_{n=0}^{\infty} 2a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} 4a_{n-1}x^n = 0$$

$$\sum_{n=1}^{\infty}\left[a_{n+1}(n+1)n - 2a_{n+2}(n+2)(n+1) + 4a_{n-1}\right]x^n - 2a_2(2)(1)(1) = 0$$

$a_2 = 0$

$$a_{n+2} = \frac{a_{n+1}(n+1)n + 4a_{n-1}}{2(n+2)(n+1)} = \frac{a_{n+1}\cdot n}{2(n+2)} + \frac{2a_{n-1}}{(n+2)(n+1)}$$

$n=1$

$$a_3 = \frac{\cancel{a_2}\cdot 1}{\cancel{2}(3)} + \frac{\cancel{2}a_0}{(3)(\cancel{2})} = \frac{1}{3}a_0$$

$n=2$

$$a_4 = \frac{a_3\cdot\cancel{2}}{\cancel{2}(4)} + \frac{2a_1}{4(3)} = \frac{1}{4}\left(\frac{1}{3}a_0\right) + \frac{1}{6}a_1 = \frac{1}{12}a_0 + \frac{1}{6}a_1$$

$n=3$

$$a_5 = \frac{a_4(3)}{2(5)} + \frac{2\cancel{a_2}^{0}}{(\cancel{5})(4)} = \frac{3}{10}\left(\frac{1}{12}a_0 + \frac{1}{6}a_1\right) = \frac{1}{40}a_0 + \frac{1}{20}a_1$$

$n=4$

$$a_6 = \frac{a_5(4)}{2(6)} + \frac{2a_3}{6\cdot 5} = \frac{1}{3}\left(\frac{1}{40}a_0 + \frac{1}{20}a_1\right) + \frac{1}{15}\left(\frac{1}{3}a_0\right) = \frac{11}{360}a_0 + \frac{1}{60}a_1$$

$$y(x) = a_0\left(1 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{40}x^5 + \frac{11}{360}x^6 + \dots\right) + a_1\left(x + \frac{1}{6}x^4 + \frac{1}{20}x^5 + \frac{1}{60}x^6 + \dots\right)$$

16. Find the general solution to the ODEs using the characteristic (or auxiliary) equation. (8 points each)

a. $y'' + \frac{2}{3}y' + \frac{1}{9}y = 0$

$r^2 + \frac{2}{3}r + \frac{1}{9} = 0 \;) \cdot 9$

$9r^2 + 6r + 1 = 0$

$(3r+1)^2 = 0$

$-\frac{1}{3} = r$ repeated

$y(t) = c_1 e^{-1/3 t} + c_2 t e^{-1/3 t}$

b. $y'' + y = 0$

$r^2 + 1 = 0$

$r = \pm i$

$y(t) = c_1 \cos t + c_2 \sin t$

17. The table below gives the solution to the second order constant coefficient homogeneous equation, and the forcing function. Determine the Ansatz for the method of undetermined coefficients in each case. (3 points each)

	y_1	y_2	y_3	$g(t)$	Ansatz
a.	e^t	te^t	1	$12t + e^t$	$At + B + Ct^2e^t$
b.	$\sin 3t$	$\cos 3t$	e^{-t}	$\sin t$	$A\sin t + B\cos t$
c.	1	x	e^x	x^3	$x^2(Ax^3 + Bx^2 + Cx + D)$
d.	$e^{-t}\sin 2t$	$e^{-t}\cos 2t$	NA	$7\cos 2t$	$A\cos 2t + B\sin 2t$

c (multiplied out) $= Ax^5 + Bx^4 + Cx^3 + Dx^2$

18. An electrical circuit has an inductance of $L = 10H$ and a capacitance of $C = 0.02F$. Find the resistance on the circuit (in ohms) needed to achieve critical damping. (8 points)

$$10Q'' + RQ' + \frac{1}{.02}Q = 0$$

$$10Q'' + RQ' + 50Q = 0$$

$$\frac{-R \pm \sqrt{R^2 - 4(10)(50)}}{20}$$

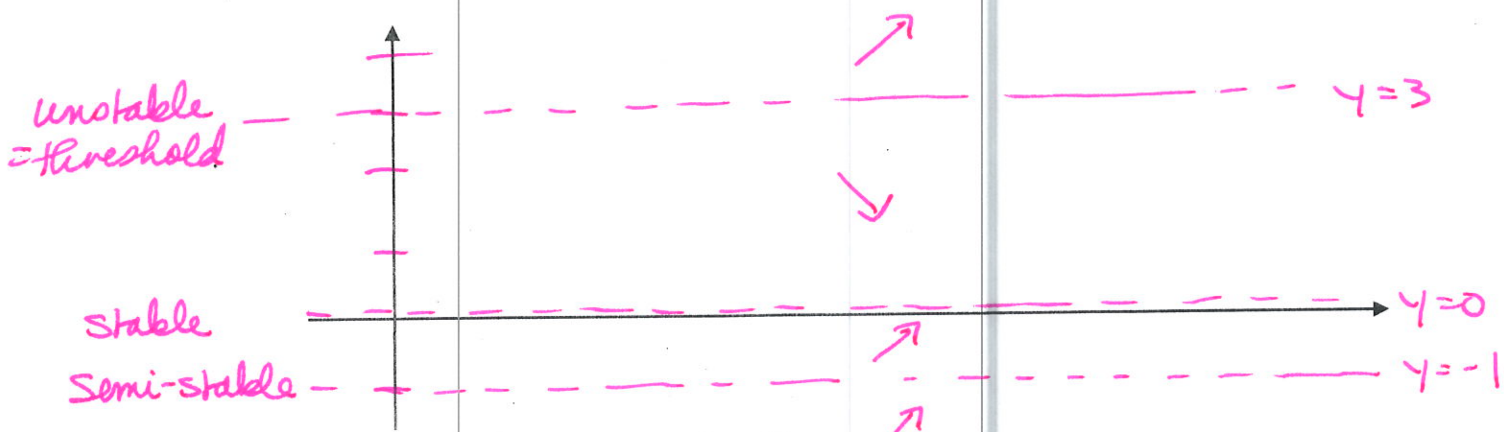
$$R^2 - 2000 = 0$$

$$R^2 = 2000$$

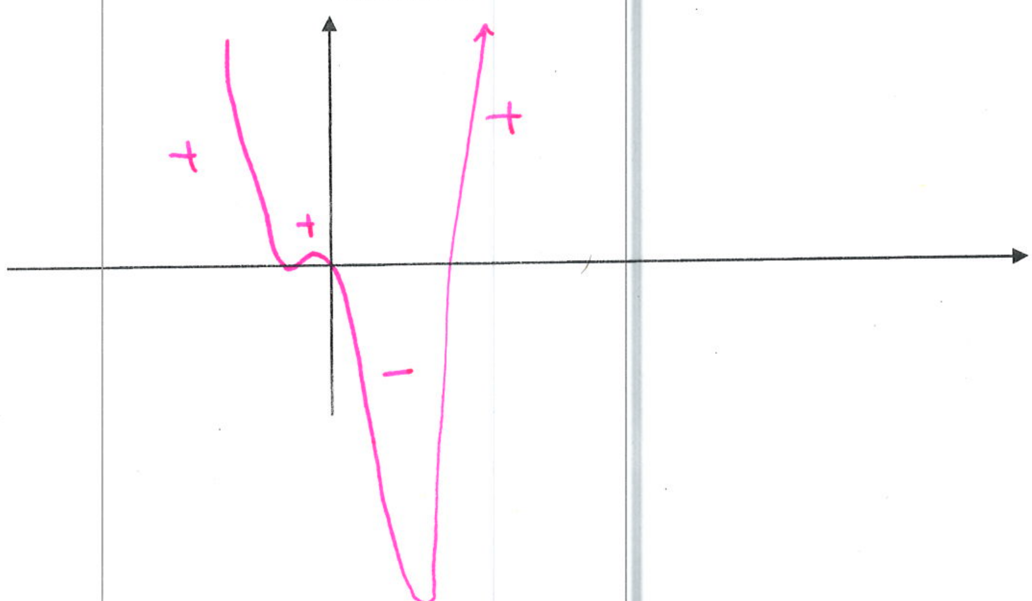
$$R = 20\sqrt{5} \approx 44.72\,\Omega$$

19. Sketch the slope field for the autonomous equation $y' = y(y+1)^2(y-3)$. Label each equilibrium and classify it as stable, semi-stable or unstable. Are any of them a carrying capacity or threshold? Draw the phase plane. (10 points)

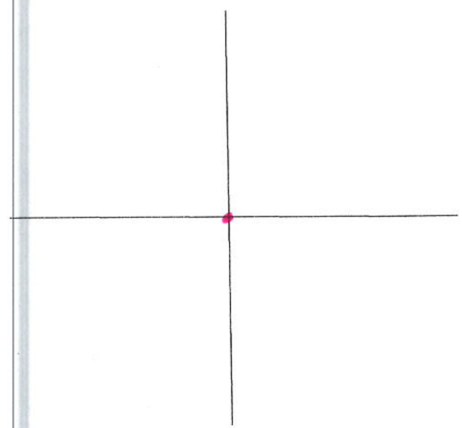
Direction Field



Phase Plane



20. Sketch the region in the plane where a solution to the ODE $\frac{dy}{dt} = \sqrt{y^2 + t^2}$ is guaranteed to exist. Be sure to check all conditions and show your work. (6 points)



$y^2+t^2 \geq 0 \qquad f = (y^2+t^2)^{1/2}$

$\frac{\partial f}{\partial y} = \frac{1}{2}(y^2+t^2)^{-1/2} \cdot 2y = \frac{y}{\sqrt{y^2+t^2}}$

$y^2+t^2 > 0$

not defined at (0,0)
defined everywhere else

21. Classify each differential equation as i) linear or nonlinear, ii) ordinary or partial, iii) its order. (3 points each)

a. $\frac{d^4y}{dx^4} = y\cos x$

4 order, ordinary, linear

b. $u_{xy} + u_{xx} = e^u \tan x \sec y$

2nd order, non linear, partial

22. A 500L tank initially contains only pure water. A hose begins adding to the tank at a rate of 5L/min with a concentration of iodine salt of 50g/L. The well-mixed solution flows out of the tank at a rate of 5L/min. Find an equation that models the amount of iodine in the tank after time t. Find the maximum amount of iodine in the tank (if one exists). (8 points)

$A(0)=0$

$\frac{dA}{dt} = \frac{5L}{min} \cdot \frac{50g}{L} - \frac{A}{500L} \cdot \frac{5L}{min}$

$A' = 250 - \frac{A}{100} = -\frac{1}{100}(A - 25{,}000)$

$\int \frac{dA}{A-25{,}000} = \int -\frac{1}{100}\,dt \Rightarrow \ln|A-25000| = -\frac{1}{100}t + C$

$A = 25000 + A_0 e^{-\frac{1}{100}t} \Rightarrow A(t) = 25000 - 25000e^{-\frac{1}{100}t}$

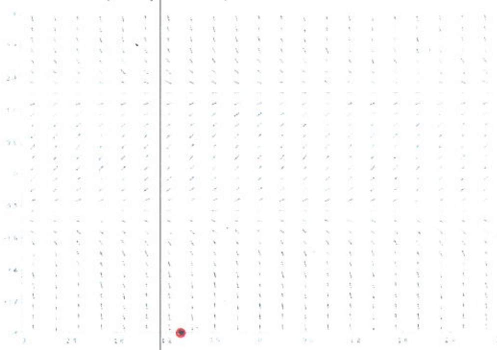
$0 = 25000 + A_0(1)$

$A_0 = 25000$

max = 25,000 g

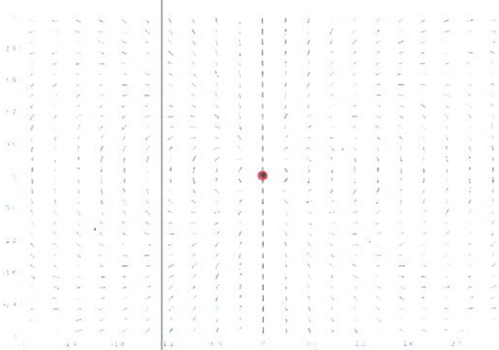
23. For each of the direction fields shown below, match the field to the differential equation that produced it. (12 points)

a.

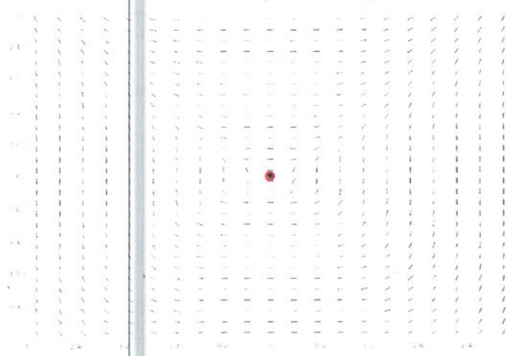


d.

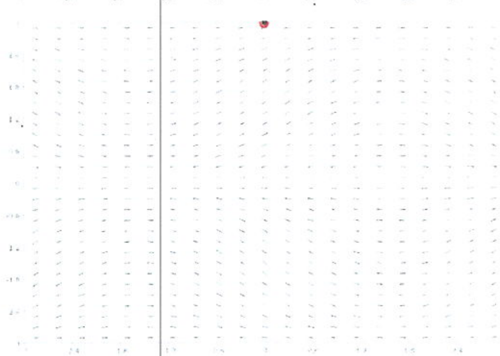
b.



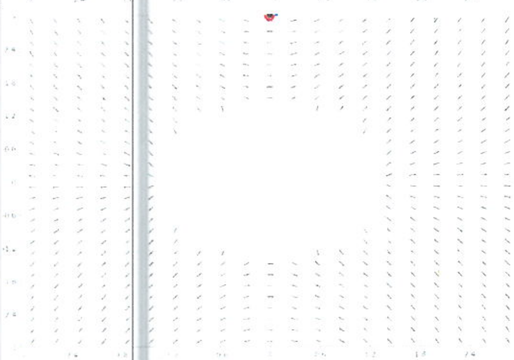
e.



c.



f.



i. $y' = \frac{t^3}{y^2}$ E

ii. $y' = \frac{ty}{\sqrt{t^2+y^2-2}}$ F

iii. $y' = -(y+1)(y-2)$ A

iv. $y' = -\sqrt{t+y^2}$ D

v. $y' = (\cos t)(\sin y)$ C

vi. $y' = \frac{y^2-t^2}{ty}$ B

24. Use the method of integrating factors to find the particular solution for $xy' = 2y + x^4e^{-x}$, $y(\ln 2) = 0$. (8 points)

$$\frac{xy' - 2y = x^4 e^{-x}}{x}$$

$$y' - \frac{2}{x}y = x^3 e^{-x}$$

$$\mu = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\cancel{\ln} x^{-2}} = x^{-2}$$

$$* x^{-2}$$

$$x^{-2}y' - 2x^{-3}y = xe^{-x}$$

$$\int (x^{-2}y)' = \int xe^{-x}\,dx$$

+/-	u	dv
+	x	e^{-x}
−	1	$-e^{-x}$
+	0	e^{-x}

$$x^{-2}y = -xe^{-x} - e^{-x} + C$$

$$y = -x^3e^{-x} - x^2e^{-x} + Cx^2$$

$$0 = -(\ln 2)^3 e^{-\ln 2} - (\ln 2)^2 e^{-\ln 2} + C(\ln 2)^2$$

$$0 = \frac{-(\ln 2)^3 \cdot \frac{1}{2} - (\ln 2)^2\left(\frac{1}{2}\right) + C(\ln 2)^2}{(\ln 2)^2}$$

$$0 = -(\ln 2)\cdot\frac{1}{2} - \frac{1}{2} + C$$

$$C = \frac{\ln 2}{2} + \frac{1}{2} = \left(\frac{\ln 2 + 1}{2}\right)$$

$$y(x) = -x^3e^{-x} - x^2e^{-x} + \left(\frac{\ln 2 + 1}{2}\right)x^2$$

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{\left(s^2+a^2\right)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{\left(s^2+a^2\right)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{\left(s^2+a^2\right)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{\left(s^2+a^2\right)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s\left(s^2-a^2\right)}{\left(s^2+a^2\right)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s\left(s^2+3a^2\right)}{\left(s^2+a^2\right)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Numerical Solutions Formula Sheet

Euler's $y_{n+1} = y_n + \Delta t\, f(t_n, y_n)$

Error in Euler' $|y_n - y(t_n)| \leq C\Delta t$

Improved Euler's

$$k_1 = f(t_n, y_n)$$
$$u_{n+1} = y_n + \Delta t\, k_1$$
$$k_2 = f(t_{n+1}, u_{n+1})$$
$$y_{n+1} = y_n + \frac{1}{2}\Delta t(k_1 + k_2)$$

Error in Improved Euler's $|y(t_n) - y_n| \leq C\Delta t^2$

Runge-Kutta

$$y_{n+1} = y_n + h\left(\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}\right)$$
$$k_{n1} = f(t_n, y_n)$$
$$k_{n2} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right)$$
$$k_{n3} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right)$$
$$k_{n4} = f(t_n + h, y_n + hk_{n3})$$

Error in Runge-Kutta $|y(t_n) - y_n| \leq C\Delta t^4$

Laplace transforms – Table

$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \quad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
e^{-at}	$\frac{1}{s + a}$	$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
te^{-at}	$\frac{1}{(s + a)^2}$	$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$\frac{1}{2}t^2 e^{-at}$	$\frac{1}{(s + a)^3}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
$\frac{1}{(n-1)!} t^{n-1} e^{-at}$	$\frac{1}{(s + a)^n}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
e^{at}	$\frac{1}{s - a} \quad s > a$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2} \quad s > \lvert\omega\rvert$
te^{at}	$\frac{1}{(s - a)^2}$	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2} \quad s > \lvert\omega\rvert$
$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s + a)(s + b)}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$
$\frac{1}{a^2}[1 - e^{-at}(1 + at)]$	$\frac{1}{s(s + a)^2}$	$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}} \quad n = 1,2,3..$	$e^{at} \sin \omega t$	$\frac{\omega}{(s - a)^2 + \omega^2}$
$t^n e^{at}$	$\frac{n!}{(s - a)^{n+1}} \quad s > a$	$e^{at} \cos \omega t$	$\frac{s - a}{(s - a)^2 + \omega^2}$
$t^n e^{-at}$	$\frac{n!}{(s + a)^{n+1}} \quad s > a$	$1 - e^{-at}$	$\frac{a}{s(s + a)}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s + a)}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}} \quad s > 0$	$f(t - t_1)$	$e^{-t_1 s} F(s)$
$g(t) \cdot p(t)$	$G(s) \cdot P(s)$	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$\int f(t)dt$	$\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$	$\delta(t)$ *unit impulse*	$1 \quad$ *all s*
$\frac{df}{dt}$	$sF(s) - f(0)$	$\frac{d^2 f}{df^2}$	$s^2 F(s) - sf(0) - f'(0)$
$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{n-1}(0)$		