212 Homework #11 Key

1. $f(t) = \begin{cases} 0 & t<1 \\ t^2-2t+2 & t\geq 1 \end{cases}$ $\quad t^2-2t+2 = (t-1)^2+1$

$[(t-1)^2+1]u_1(t)$ or $[(t-1)^2+1]u(t-1)$

$\Rightarrow \mathcal{L}\{f(t)\} = e^{-s}\left(\frac{2}{s^3}+\frac{1}{s}\right)$

2a. $\frac{3!}{(s-2)^4} = t^3e^{2t}$

b. $\frac{(s-2)e^{-s}}{s^2-4s+3} = e^{-s}\left[\frac{s-2}{(s-3)(s-1)}\right] = e^{-s}\left[\frac{A}{s-3}+\frac{B}{s-1}\right]$

$As-A+Bs-3B$

$A+B=1$

$-A-3B=-2$ $\quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -3 & -2 \end{array}\right]$ $\quad A=1/2$, $B=1/2$

$\frac{1}{2}e^{-s}\left[\frac{1}{s-3}+\frac{1}{s-1}\right] = \frac{1}{2}u_1(t)\left[e^{3(t-1)}+e^{t-1}\right]$

c. $\frac{e^{-2s}}{s^2-4} = \frac{e^{-2s}}{2}\left[\frac{2}{s^2-4}\right] = \frac{1}{2}u_2(t)\sinh(2(t-2))$

$= \frac{1}{2}u_2(t)\sinh(2t-4)$

3. a. $\mathcal{L}\{f(t)\}$ for $f(t) = \int_0^t (t-\tau)^2\cos 2\tau\, d\tau$

$g=t^2 \quad h=\cos 2t$

$\mathcal{L}\{f(t)\} = \frac{2}{s^3}\cdot\frac{s}{s^2+4} = \frac{2}{s^2(s^2+4)}$

b. $\mathcal{L}^{-1}\{F(s)\}$ $\quad F(s) = \frac{s}{(s+1)(s^2+4)} = \frac{1}{s+1}\cdot\frac{s}{s^2+4}$

$f(t) = \int_0^t e^{-(t-\tau)}\sin(2\tau)\,d\tau$

4. a. $y''+2y'+2y=\sin\alpha t$ $\quad y(0)=0,\ y'(0)=0$

$s^2Y(s)+2sY(s)+2Y(s) = \frac{\alpha}{s^2+\alpha^2}$

4a (cont'd)

$$Y(s) = \frac{\alpha}{s^2+\alpha^2} \cdot \frac{1}{s^2+2s+2} = \frac{\alpha}{s^2+\alpha^2} \cdot \frac{1}{(s+1)^2+1}$$

$$y(t) = \int_0^t \sin(\alpha(t-\tau))\, e^{-\tau} \sin\tau \, d\tau$$

$$\text{or } \int_0^t e^{-t+\tau} \sin(t-\tau) \sin(\alpha\tau)\, d\tau$$

4b. $y^{IV} - y = g(t)$

$$s^4 Y(s) - Y(s) = G(s) \Rightarrow Y(s) = G(s) \cdot \frac{1}{s^4-1} = G(s) \frac{1}{(s^2+1)(s^2-1)}$$

$$\frac{A\cancel{s}+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+1}$$

$$(As+B)(s^2-1) + C(s+1)(s^2+1) + D(s-1)(s^2+1)$$

$$As^3 - As + Bs^2 - B + C(s^3+s+s^2+1) + D(s^3+s-s^2-1)$$

$$As^3 - As + Bs^2 - B + Cs^3 + Cs + Cs^2 + C + Ds^3 + Ds - Ds^2 - D$$

$A + C + D = 0$
$B + C - D = 0$
$-A - B + C + D = 0$
$-B + C - D = 1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 1 \end{array}\right]$$

$A = 1/4$
$B = -1/2$
$C = 1/8$
$D = -3/8$

$$Y(s) = G(s)\left[\frac{1}{4}\left(\frac{s}{s^2+1}\right) - \frac{1}{2}\left(\frac{1}{s^2+1}\right) + \frac{1}{8}\left(\frac{1}{s-1}\right) - \frac{3}{8}\left(\frac{1}{s+1}\right)\right]$$

$$y(t) = \int_0^t g(t-\tau)\left[\frac{1}{4}\cos\tau - \frac{1}{2}\sin\tau + \frac{1}{8}e^{t} - \frac{3}{8}e^{-t}\right]$$

5. $\varphi'(t) + \varphi(t) = \int_0^t \sin(t-\xi)\,\varphi(\xi)\,d\xi, \ \varphi(0) = 1$ $\qquad \mathcal{L}\{\varphi(t)\} = \Phi(s)$

$$\underbrace{s\Phi(s)}_{-s} + \Phi(s) = \frac{1}{s^2+1} \cdot \Phi(s) \Rightarrow s\Phi(s) + \Phi(s) - \frac{1}{s^2+1}\Phi(s) = s$$

$$\Phi(s)\left[s + 1 - \frac{1}{s^2+1}\right] = s$$

$$\frac{s^3+s+s^2+1-1}{s^2+1} = \frac{s^3+s^2+s}{s^2+1}$$

5 cont'd

$$\Phi(s) = \frac{s(s^2+1)}{s^3+s^2+s} = \frac{\cancel{s}(s^2+1)}{\cancel{s}(s^2+s+1)} = \frac{s^2+1}{s^2+s+1} =$$

$$1 - \frac{s}{s^2+s-1} = 1 - \frac{s}{(s^2+s+1/4)+3/4} =$$

$$\begin{array}{r} 1 \\ s^2+s+1 \enclose{longdiv}{s^2+0s+1} \\ -\underline{s^2 \mp s \mp 1} \\ -s \end{array}$$

$$1 - \frac{s}{(s+1/2)^2+\frac{3}{4}} = 1 - \left[\frac{s+1/2-1/2}{(s+1/2)^2+\frac{3}{4}}\right] =$$

$$1 - \frac{(s+1/2)}{(s+1/2)^2+\frac{3}{4}} + \frac{1/2}{(s+1/2)^2+\frac{3}{4}} \cdot \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{3}}$$

$a^2 = \frac{3}{4}$

$a = \frac{\sqrt{3}}{2}$

$$= 1 - \frac{(s+1/2)}{(s+1/2)+\frac{3}{4}} + \frac{1}{\sqrt{3}}\left(\frac{(\sqrt{3}/2)}{(s+1/2)^2+\frac{3}{4}}\right) = \Phi(s)$$

$$\varphi(t) = \delta(t) - e^{-1/2t}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}e^{-1/2t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

6a. $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$ $\quad y(0)=0,\ y'(0)=0$

$$s^2Y(s) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}$$

$$Y(s)(s^2+4) = e^{-\pi s} - e^{-2\pi s} \Rightarrow Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s^2+4}$$

$$e^{-\pi s}\left(\frac{1}{s^2+4}\right) - e^{-2\pi s}\left(\frac{1}{s^2+4}\right)$$

$$y(t) = u_\pi(t)\sin(2(t-\pi)) - u_{2\pi}(t)\sin(2(t-2\pi))$$

$$= u_\pi(t)\sin(2t-\cancel{2\pi}) - u_{2\pi}(t)\sin(2t-\cancel{4\pi})$$ cancels since periodic

$$= u_\pi(t)\sin 2t - u_{2\pi}(t)\sin(2t)$$

$$\begin{cases} 0 & t<\pi \\ \sin 2t & \pi \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases}$$

b. $y'' + y = \delta(t-2\pi) + \cos t$ $\quad y(0)=0 \quad y'(0)=1$

$$s^2Y(s) - 1 + Y(s) = e^{-2\pi s} + \frac{s}{s^2+1}$$

$$Y(s)(s^2+1) = 1 - e^{-2\pi s} + \frac{s}{s^2+1}$$

6b cont'd

$$Y(s) = \frac{1}{s^2+1} - e^{-2\pi s}\left(\frac{1}{s^2+1}\right) + \frac{s}{(s^2+1)^2}$$

$$y(t) = \sin(t) - \underbrace{u_{2\pi}\sin t}_{=(\sin(t-2\pi))} + \tfrac{1}{2}t\sin t$$

$$= \begin{cases} \sin t + \tfrac{1}{2}t\sin t & t < 2\pi \\ \tfrac{1}{2}t\sin t & t \geq 2\pi \end{cases}$$

6c. $y^{iv} - y = \delta(t-1)$, $y(0)=0$, $y'(0)=0$, $y''(0)=0$, $y'''(0)=0$

$$s^4Y(s) - Y(s) = e^{-s}$$

$$(s^4-1)Y(s) = e^{-s} \Rightarrow Y(s) = e^{-s}\left(\frac{1}{\underset{(s^2+1)(s^2-1)}{s^4-1}}\right) = e^{-s}\left(\frac{1}{2}\cdot\frac{1}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2-1}\right)$$

$$\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-1} =$$

$$As^3 + Bs^2 - As - B + Cs^3 + Ds^2 + Cs + D$$

$A + C = 0$ $\longrightarrow$ $2C = 0 \Rightarrow C = 0 \Rightarrow A = 0$

$B + D = 0$

$-A + C = 0$ $\longrightarrow$ $2D = 1$

$-B + D = 1$

$D = 1/2$ $B = 1/2$

$$y(t) = \tfrac{1}{2}u_1(t)\left(\sin(t-1) + \sinh(t-1)\right)$$

7a. $\begin{bmatrix} 1 & 6 \\ 2 & 5 \end{bmatrix}$ $(1-\lambda)(5-\lambda) - 12$

$\lambda^2 - 6\lambda + 5 - 12 = 0$

$\lambda^2 - 6\lambda - 7 = 0$

$(\lambda - 7)(\lambda + 1) = 0$

$\lambda = 7$, $\lambda = -1$

$\begin{bmatrix} -6 & 6 \\ 2 & -2 \end{bmatrix} \xrightarrow[\div 2]{} x_1 - x_2 = 0$

$x_1 = x_2$

$x_2 = x_2$ $\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda = 7$

$\begin{matrix} 2 & 6 \\ 2 & 6 \end{matrix} \xrightarrow[\div 2]{} x_1 + 3x_2 = 0$

$x_1 = -3x_2$ $\vec{V}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$x_2 = x_2$ $\lambda = -1$

b. $\begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$ $\lambda = -2$, $\lambda = 2$

$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ $x_1 = -x_2$, $x_2 = x_2$ $\vec{V}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$(-1-\lambda)(1-\lambda) - 3 = 0$

$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$ $3x_1 = x_2$, $x_2 = x_2$ $\vec{V}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda^2 - 1 - 3 =$

$\lambda^2 - 4 = 0$

6c. $\begin{bmatrix}-4 & 1\\ 6 & -5\end{bmatrix}$ $\lambda=7, \lambda=-2$ $\begin{bmatrix}3 & 1\\ 6 & 2\end{bmatrix}$ $3x_1=-x_2$ $x_2=x_2$ $\vec{v}_1=\begin{bmatrix}-1\\ 3\end{bmatrix}$ $\begin{bmatrix}-2 & 1\\ 6 & -3\end{bmatrix}$ $-2x_1=-x_2$ $x_2=x_2$ $\vec{v}_2=\begin{bmatrix}1\\ 2\end{bmatrix}$

$(\lambda+7)(\lambda+2)=0$

$(-4-\lambda)(-5-\lambda)-6$

$\lambda^2+9\lambda+20-6$

$\lambda^2+9\lambda+14=0$

6d. $\begin{bmatrix}-2 & 2\\ -5 & 6\end{bmatrix}$ $\lambda_1=2+\sqrt{6}$ $\lambda_2=2-\sqrt{6}$

$(-2-\lambda)(6-\lambda)+10$ $\begin{bmatrix}-2-2-\sqrt{6} & 2\\ -5 & 6-2-\sqrt{6}\end{bmatrix}$ $-5x_1+(4-\sqrt{6})x_2=0$ $\vec{v}_1=\begin{bmatrix}4-\sqrt{6}\\ 5\end{bmatrix}$ $\vec{v}_2=\begin{bmatrix}4+\sqrt{6}\\ 5\end{bmatrix}$

$\lambda^2+4\lambda-2=0$ $x_1=\dfrac{-4+\sqrt{6}}{-5}x_2$

$\dfrac{-4\pm\sqrt{16+8}}{2}=2\pm\sqrt{6}$ $\begin{bmatrix}-4-\sqrt{6} & 2\\ -5 & 4-\sqrt{6}\end{bmatrix}$ $x_2=x_2$

6e. $\begin{bmatrix}2 & 9\\ 1 & 10\end{bmatrix}$ $\lambda=11, \lambda=1$

$(2-\lambda)(10-\lambda)-9$ $\begin{bmatrix}-9 & 9\\ 1 & -1\end{bmatrix}$ $x_1=x_2$ $x_2=x_2$ $\vec{v}_1=\begin{bmatrix}1\\ 1\end{bmatrix}$ $\begin{bmatrix}1 & 9\\ 1 & 9\end{bmatrix}$ $x_1=-9x_2$ $x_2=x_2$ $\vec{v}_2=\begin{bmatrix}-9\\ 1\end{bmatrix}$

$\lambda^2-12\lambda-11=0$

$(\lambda-11)(\lambda-1)=0$

f. $\begin{bmatrix}-2 & 5\\ 7 & 0\end{bmatrix}$ $\lambda=-7, \lambda=5$

$\begin{bmatrix}-2 & -2\\ 7 & 7\end{bmatrix}$ $\div 7$ $x_1=-x_2$ $x_2=x_2$ $\vec{v}_1=\begin{bmatrix}-1\\ 1\end{bmatrix}$ $\begin{bmatrix}-7 & 5\\ 7 & -5\end{bmatrix}$ $7x_1=5x_2$ $x_2=x_2$ $\vec{v}_2=\begin{bmatrix}5\\ 7\end{bmatrix}$

$(-2-\lambda)(-\lambda)-35$

$\lambda^2+2\lambda-35=0$

$(\lambda+7)(\lambda-5)=0$

g. $\begin{bmatrix}3 & -2\\ 2 & 3\end{bmatrix}$ $\dfrac{6\pm\sqrt{36-52}}{2}$ $\begin{bmatrix}3-3-2i & -2\\ 2 & 3-3-2i\end{bmatrix}=\begin{bmatrix}-2i & -2\\ 2 & -2i\end{bmatrix}$ $\cancel{2}x_1=+\cancel{2}x_2$ $x_2=x_2$

$(3-\lambda)^2+4=0$ $\lambda=3\pm 2i$ $\vec{v}_1=\begin{bmatrix}i\\ 1\end{bmatrix}$ $\vec{v}_2=\begin{bmatrix}-i\\ 1\end{bmatrix}$

$\lambda^2-6\lambda+9+4=0$

$\lambda^2-6\lambda+13=0$

h. $\begin{bmatrix}-2 & 5 & 3\\ 0 & 2 & -4\\ 0 & -1 & 2\end{bmatrix}$ $\lambda_1=-2, \lambda_2=0, \lambda_3=4$

$\begin{bmatrix}0 & 5 & 3\\ 0 & 4 & -4\\ 0 & -1 & 4\end{bmatrix}$ rref $x_1=x_1$ $\boxed{\begin{matrix}x_2=4x_3\\ 4x_2=4x_2\end{matrix}}$ $\Rightarrow x_2=0$ $x_3=0$ $\vec{v}_1=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$

$\begin{bmatrix}-6 & 5 & 3\\ 0 & -2 & -4\\ 0 & -1 & -2\end{bmatrix}$ rref $x_3=x_3$ $x_2=2x_3$ $-6x_1=-5x_2-3x_3$ $\vec{v}_3=\begin{bmatrix}-5\\ -12\\ 6\end{bmatrix}$

$(-2-\lambda)\begin{vmatrix}2-\lambda & -4\\ -1 & 2-\lambda\end{vmatrix}$

$(-2-\lambda)\left[(2-\lambda)(2-\lambda)-4\right]$

$\lambda=-2$

$\lambda^2-4\lambda+4-4$

$\lambda^2-4\lambda=0$

$\lambda=0$ $\lambda=4$

$\begin{bmatrix}-2 & 5 & 3\\ 0 & 2 & -4\\ 0 & -1 & 2\end{bmatrix}$ rref $x_2=2x_3$ $x_3=x_3$ $-2x_1=-5x_2-3x_3$ $\vec{v}_2=\begin{bmatrix}15\\ 4\\ 2\end{bmatrix}$

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6i. $\begin{bmatrix} 4 & 5 \\ 6 & 11 \end{bmatrix}$

$(4-\lambda)(11-\lambda)-30$

$\lambda^2-15\lambda+44-30$

$\lambda^2-15\lambda+14=0$

$(\lambda-14)(\lambda-1)$

$\lambda=14$

$\lambda=1$

$\begin{bmatrix} -10 & 5 \\ 6 & -3 \end{bmatrix} \xrightarrow{\div 3}$ $2x_1=x_2$ $\quad x_2=x_2$ $\quad \vec{v}_1=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 3 & 5 \\ 6 & 10 \end{bmatrix}$ $3x_1=-5x_2$ $\quad x_2=x_2$ $\quad \vec{v}_2=\begin{bmatrix} -5 \\ 3 \end{bmatrix}$

6j. $\begin{bmatrix} -3 & 7 \\ 5 & -1 \end{bmatrix}$

$(-3-\lambda)(-1-\lambda)-35$

$\lambda^2+4\lambda+\cancel{3}-35$

$\lambda^2+4\lambda-32=0$

$(\lambda+8)(\lambda-4)=0$

$\lambda=-8,\ \lambda=4$

$\begin{bmatrix} 5 & 7 \\ 5 & 7 \end{bmatrix}$ $5x_1=-7x_2$ $\quad x_2=x_2$ $\quad \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

$\begin{bmatrix} -7 & 7 \\ 5 & -5 \end{bmatrix} \xrightarrow{\div 5}$ $x_1=x_2$ $\quad x_2=x_2$ $\quad \vec{v}_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

6k. $\begin{bmatrix} -4 & 5 \\ -5 & -4 \end{bmatrix}$

$(-4-\lambda)^2+25=0$

$\lambda^2+8\lambda+16+25=0$

$\lambda^2+8\lambda+41=0$

$\lambda=\dfrac{-8\pm\sqrt{64-164}}{2}$

$\dfrac{-8\pm 10i}{2}=-4\pm 5i$

$\begin{bmatrix} -4+4-5i & 5 \\ -5 & -4+4-5i \end{bmatrix}$

$\begin{bmatrix} -5i & 5 \\ -5 & -5i \end{bmatrix}$

$-5x_1=5ix_2$

$x_2=x_2$

$\vec{v}_1=\begin{bmatrix} -i \\ 1 \end{bmatrix},\ \vec{v}_2=\begin{bmatrix} i \\ 1 \end{bmatrix}$