212 Homework #6 Key

1.a. $\begin{cases} 2x+3y=12 \\ 4x-y=10 \end{cases}$

$A=\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ $\det A = -2-12 = -14$

$A_1 = \begin{bmatrix} 12 & 3 \\ 10 & -1 \end{bmatrix} \Rightarrow \det A_1 = -12-30 = -42$

$A_2 = \begin{bmatrix} 2 & 12 \\ 4 & 10 \end{bmatrix} \Rightarrow \det A_2 = 20-48 = -28$

$x_1 = \dfrac{\det A_1}{\det A} = \dfrac{-42}{-14} = 3$ $x_2 = \dfrac{\det A_2}{\det A} = \dfrac{-28}{-14} = 2$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

b. $\begin{cases} -x+5y=17 \\ 3x-4y=12 \end{cases}$

$A = \begin{bmatrix} -1 & 5 \\ 3 & -4 \end{bmatrix} \Rightarrow \det A = 4-15 = -11$

$A_1 = \begin{bmatrix} 17 & 5 \\ 12 & -4 \end{bmatrix} \Rightarrow \det A_1 = -68-60 = -128$

$A_2 = \begin{bmatrix} -1 & 17 \\ 3 & 12 \end{bmatrix} \Rightarrow \det A_2 = -12-51 = -63$

$x_1 = \dfrac{\det A_1}{\det A} = \dfrac{-128}{-11} = \dfrac{128}{11}$ $x_2 = \dfrac{\det A_2}{\det A} = \dfrac{-63}{-11} = \dfrac{63}{11}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 128/11 \\ 63/11 \end{bmatrix}$

c. $\begin{cases} 5x-y+2z=10 \\ 3x+2y-4z=16 \\ -4x-3y+z=7 \end{cases}$

$A = \begin{bmatrix} 5 & -1 & 2 \\ 3 & 2 & -4 \\ -4 & -3 & 1 \end{bmatrix}$ $\det A = -65$

$A_1 = \begin{bmatrix} 10 & -1 & 2 \\ 16 & 2 & -4 \\ 7 & -3 & 1 \end{bmatrix}$ $\det A_1 = -180$

$A_2 = \begin{bmatrix} 5 & 10 & 2 \\ 3 & 16 & -4 \\ -4 & 7 & 1 \end{bmatrix}$ $\det A_2 = 520$

$A_3 = \begin{bmatrix} 5 & -1 & 10 \\ 3 & 2 & 16 \\ -4 & -3 & 7 \end{bmatrix}$ $\det A_3 =$

$x_1 = \dfrac{\det A_1}{\text{let} A} = \dfrac{-180}{-65} = \dfrac{36}{13}$ $x_2 = \dfrac{\det A_2}{\det A} = \dfrac{520}{-65} = -8$ $x_3 = \dfrac{\det A_3}{\det A} = \dfrac{385}{-65} = -\dfrac{77}{13}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 36/13 \\ -8 \\ -77/13 \end{bmatrix}$

1d. $\begin{cases} x+y+z=9 \\ -x+2y-3z=14 \\ 3x-5y-2z=-18 \end{cases}$ $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -3 \\ 3 & -5 & -2 \end{bmatrix}$ $\det A = -31$

$A_1 = \begin{bmatrix} 9 & 1 & 1 \\ 14 & 2 & -3 \\ -18 & -5 & -2 \end{bmatrix}$ $\det A_1 = -123$

$A_2 = \begin{bmatrix} 1 & 9 & 1 \\ -1 & 14 & -3 \\ 3 & -18 & -2 \end{bmatrix}$ $\det A_2 = -205$

$A_3 = \begin{bmatrix} 1 & 1 & 9 \\ -1 & 2 & 14 \\ 3 & -5 & -18 \end{bmatrix}$ $\det A_3 = 49$

$x_1 = \frac{\det A_1}{\det A} = \frac{-123}{-31} = \frac{123}{31}$ $x_2 = \frac{\det A_2}{\det A} = \frac{-205}{-31} = \frac{205}{31}$ $x_3 = \frac{\det A_3}{\det A} = \frac{49}{-31} = \frac{-49}{31}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 123/31 \\ 205/31 \\ -49/31 \end{bmatrix}$

2a. $t^2y'' + 2ty' - 2y = 0$ $y_1(t) = t$ $y_2 = vt$ $y_2' = v't + v$ $y_2'' = v'' + 2v'$

$t^2(v''+2v') + 2t(v't+v) - 2vt = 0$

$t^2v'' + 2v't^2 + 2t^2v' + \cancel{2vt} \cancel{-2vt} = 0 \Rightarrow t^2(v''+4v') = 0$

$u = v'$ $u' = v''$

$u' = -4u \Rightarrow \int \frac{du}{u} = \int -4\,dt$

$\ln u = -4t + C \Rightarrow u = Ae^{-4t} \Rightarrow v' = Ae^{-4t} \Rightarrow v = \frac{-A}{4}e^{-4t} = Be^{-4}$

$y_2 = e^{-4t} \cdot t$

$\begin{vmatrix} t & te^{-4t} \\ 1 & e^{-4t} - 4te^{-4t} \end{vmatrix} = te^{-4t} - 4t^2e^{-4t} - te^{-4t} = -4t^2e^{-4t}$

Yes, fundamental set

b. $(x-1)y'' - xy' + y = 0$ $y_1 = e^x$ $y_2 = ve^x$ $y_2' = v'e^x + ve^x$

$y_2'' = v''e^x + 2v'e^x + ve^x$

$(x-1)(v''e^x + 2v'e^x + ve^x) - x(v'e^x + ve^x) + ve^x = 0$ $/e^x$

$(x-1)(v'' + 2v' + v) - x(v' + v) + v = 0 \Rightarrow xv'' + 2xv' + \cancel{vx} - v'' - 2v' \cancel{-v} - xv' - \cancel{vx} \cancel{+v} =$

$v''(x-1) + v'(x-2) = 0$ $u = v'$ $u' = v''$

$x-1 \overline{)\,-x+2}$ quotient -1, $\frac{-x+1}{1}$

$\frac{du}{dx}(x-1) = -(x-2)u \Rightarrow \frac{du}{u} = -\frac{(x-2)}{x-1} = -1 + \frac{1}{x-1}$

$\ln u = -\cancel{x} + \ln(x-1) + C \Rightarrow u = A(x-1)e^{-x} = v'$ $v = \int (x-1)e^{-x}\,dx$

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2b. (cont'd)

$$p = (x-1) \quad dq = e^{-x}$$
$$dp = dx \quad q = -e^{-x}$$

$$v = -(x-1)e^{-x} + \int e^{-x}\,dx = (1-x)e^{-x} - e^{-x} =$$
$$e^{-x} - xe^{-x} - e^{-x} = -xe^{-x}$$

$$y_2 = xe^{-x} \cdot e^{x} \Rightarrow y_2 = x \qquad W = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x - xe^x \neq 0 \text{ fundamental set}$$

3. a. $y'' + 2y' + 2y = 0 \qquad g(t) = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2 \sin t$

$$r^2 + 2r + 2 = 0$$
$$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i \qquad y(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$

$$Y(t) = Ae^{-t} + Bt^3 e^{-t}\cos t + Ct^2 e^{-t}\cos t + Dte^{-t}\cos t + Et^3 e^{-t}\sin t$$
$$+ Ft^2 e^{-t}\sin t + Gte^{-t}\sin t$$

b. $y'' + 4y = t^2 \sin 2t + (6t+7)\cos 2t$

$$y'' + 4y = 0 \Rightarrow r^2 + 4 = 0 \quad r = \pm 2i \Rightarrow y(t) = c_1 \cos 2t + c_2 \sin 2t$$

$$Y(t) = At^3 \sin 2t + Bt^2 \sin 2t + Ct \sin 2t + Dt^3 \cos 2t + Et^2 \cos 2t$$
$$+ Ft \cos 2t$$

4a. $y'' + y = \tan t \qquad y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i \quad y_1 = \cos t \quad y_2 = \sin t$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$Y(t) = -y_1 \int \frac{y_2 g}{W}\,dt + y_2 \int \frac{y_1 g}{W}\,dt = -\cos t \int \sin t \cdot \tan t\,dt + \sin t \int \cos t \tan t\,dt$$

$$= -\cos t \int \frac{\sin^2 t}{\cos t}\,dt + \sin t \int \sin t\,dt = -\cos t \int \frac{1-\cos^2 t}{\cos t}\,dt + \sin t \int \sin t\,dt$$

$$= -\cos t \int \sec t - \cos t\,dt + \sin t \cdot (-\cos t) = -\cos t\left[\ln|\sec t + \tan t| - \sin t\right]$$

$$-\cos t \sin t = -\cos t \ln|\sec t + \tan t| + \cancel{\sin t \cos t} - \cancel{\cos t \sin t} =$$

$$-\cos t \ln|\sec t + \tan t|$$

$$y(t) = c_1 \cos t + c_2 \sin t - \cos t \ln|\sec t + \tan t|$$

b. $ty'' - (1+t)y' + y = t^2 e^{2t} \qquad y_1 = 1+t \quad y_2 = e^t$

$$W = \begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = \cancel{e^t} + te^t - \cancel{e^t} = te^t$$

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4b cont'd

$$y(t) = -(1+t)\int \frac{\cancel{t}e^{2t}\cdot \cancel{e^t}}{\cancel{t}\cancel{e^t}}\,dt + e^t\int\frac{(1+t)t^{\cancel{2}}e^{\cancel{2}t}}{\cancel{t}\cancel{e^t}}\,dt =$$

$$-(1+t)\int te^{2t}\,dt + e^t\int (t+t^2)e^t\,dt$$

$\pm$	u	dv
+	t	e^{2t}
−	1	$\frac{1}{2}e^{2t}$
+	0	$\frac{1}{4}e^{2t}$

$\pm$	u	dv
+	$t+t^2$	e^t
−	$1+2t$	e^t
+	2	e^t
−	0	e^t

$$-(1+t)\left[\tfrac{1}{2}te^{2t} - \tfrac{1}{4}e^{2t}\right] + e^t\left[(t+t^2)e^t - (1+2t)e^t + 2e^t\right]$$

$$e^{2t}\left[-\tfrac{1}{2}t - \tfrac{1}{2}t^2 + \tfrac{1}{4} + \tfrac{1}{4}t + t + t^2 - 1 - 2t + 2\right] = e^{2t}\left[\tfrac{1}{2}t^2 - \tfrac{5}{4}t + \tfrac{5}{4}\right]$$

$+1$

$-\tfrac{1}{4}t$ $\quad -t$

$$y(t) = c_1(1+t) + c_2(e^t) + e^{2t}\left(\tfrac{1}{2}t^2 - \tfrac{5}{4}t + \tfrac{5}{4}\right)$$

4c. $x^2y'' - 3xy' + 4y = x^2\ln x$, $y_1 = x^2$, $y_2 = x^2\ln x$

$$W = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & 2x\ln x + x \end{vmatrix} = \cancel{2x^3\ln x} + x^3 \cancel{- 2x^3\ln x} = x^3$$

$$-x^2\int\frac{x^2\ln x\cdot x^2\ln x}{x^3}\,dx + x^2\ln x\int\frac{x^2\cdot x^2\ln x}{x^3}\,dx =$$

$$-x^2\int x\ln^2 x\,dx + x^2\ln x\int x\ln x\,dx$$

$u = \ln^2 x$ $\quad dv = x$ $\qquad u = \ln x$ $\quad dv = x$

$du = 2\ln x\cdot\frac{1}{x}\,dx$ $\quad v = \frac{1}{2}x^2$ $\qquad du = \frac{1}{x}$ $\quad v = \frac{1}{2}x^2$

$$-x^2\left[\tfrac{1}{2}x^2\ln^2 x - \int x\ln x\,dx\right] + x^2\ln x\left[\tfrac{1}{2}x^2\ln x - \int\tfrac{1}{2}x\,dx\right]$$

$$-x^2\left[\tfrac{1}{2}x^2\ln^2 x - \tfrac{1}{2}x^2\ln x + \tfrac{1}{4}x^2\right] + x^2\ln x\left[\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2\right] =$$

$$\cancel{-\tfrac{1}{2}x^4\ln^2 x} + \tfrac{1}{2}x^4\ln x - \tfrac{1}{4}x^4 + \cancel{\tfrac{1}{2}x^4\ln^2 x} - \tfrac{1}{4}x^4\ln x$$

$$\tfrac{1}{4}x^4\ln x - \tfrac{1}{4}x^4$$

$$y(t) = c_1x^2 + c_2x^2\ln x + \tfrac{1}{4}x^4\ln x - \tfrac{1}{4}x^4$$

212 Homework #6 key Cont'd

4d. $Y'' - 2y' + Y = \frac{e^t}{1+t^2}$

$Y'' - 2y' + y = 0$
$r^2 - 2r + 1 = 0$
$(r-1)^2 = 0$ $r = 1$ repeated $y_1 = e^t$, $y_2 = te^t$

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} \cancel{+te^{2t}} \cancel{-te^{2t}} = e^{2t}$$

$$-e^t \int \frac{te^t \cdot e^t}{e^{2t}(1+t^2)} dt + te^t \int \frac{e^t \cdot e^t}{e^{2t}(1+t^2)} dt = -e^t \int \frac{t}{1+t^2} dt + te^t \int \frac{1}{1+t^2} dt =$$

$$-e^t \left(\tfrac{1}{2} \ln|1+t^2|\right) + te^t \arctan t$$

$$y(t) = c_1 e^t + c_2 te^t - \tfrac{1}{2} e^t \ln|1+t^2| + te^t \arctan t$$

5. undetermined coefficients okay:

e^t, $\sin t$, $\cos t$, t^4, t^3, $e^t \sin t$, te^t

must use variation of parameters:

$\sqrt{t}$, $\tan t$, $\ln t$, $t \sec t$, $\ln(\cos t)$

6. a. $u(t) = 3\cos 2t + 4\sin 2t$ (QI)

$R = \sqrt{9+16} = \sqrt{25} = 5$ $\tan^{-1}\left(\frac{4}{3}\right) \approx 53.1°$ or $\boxed{.9273}$ $\omega = 2$

$u(t) = 5\cos(2t - .9273)$

b. $u(t) = -2\cos \pi t - 3\sin \pi t$ (QIII) $\omega = \pi$

$R = \sqrt{4+9} = \sqrt{13}$ $\tan^{-1}\left(\frac{-3}{-2}\right) \approx \boxed{.9828}$ or $56.3°$
$+\pi$

$u(t) = \sqrt{13} \cos(\pi t - 4.1244)$

8. $LQ'' + RQ' + \frac{1}{C} Q = E(t)$ $Q(0) = 10^{-6}$
$Q'(0) = 0$

$.2Q'' + 300Q' + \frac{1}{10^{-5}} Q = 0$

(x5)
$Q'' + 1500Q' + 5\times10^5 Q = 0$

$$r = \frac{-1500 \pm \sqrt{250{,}000}}{2} = -750 \pm 250 \quad r = -500, -1000$$

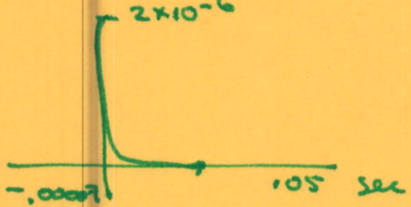
8 cont'd

$Q(t) = c_1 e^{-500t} + c_2 e^{-1000t}$

$10^{-6} = c_1 + c_2$ $\qquad C_1 = 2\times 10^{-6}$

$0 = -500c_1 - 1000c_2$ $\qquad C_2 = -1\times 10^{-6}$

$Q(t) = 2\times 10^{-6} e^{-500t} - 10^{-6} e^{-1000t}$



7. $.2Q'' + RQ' + \frac{1}{.8\times 10^{-6}} Q = 0$

$(\times 5)$ $Q'' + 5RQ' + 1.25\times 10^6 Q = 0$

$r = \frac{-5R \pm \sqrt{25R^2 + (-4)(1.25\times 10^6)}}{2}$ ← critically damped

when $25R^2 - 5\times 10^6 = 0$

$25R^2 = 5\times 10^6$

$R^2 = 2\times 10^5$ $\quad R \approx 447.21\,\Omega$

9. $k=4 \quad m=1 \quad y'' + 4y = 0 \qquad y(0) = .5,\ y'(0) = -1$

$r^2 + 4 = 0 \quad r = \pm 2i \qquad y(t) = c_1\cos 2t + c_2\sin 2t \qquad c_1(1) + \cancel{c_2(0)} = .5 \Rightarrow c_1 = \frac{1}{2}$

$y'(t) = -2c_1 \sin 2t + 2c_2\cos 2t \qquad \cancel{-(0)} + 2c_2(1) = -1$

$-\sin 2t \qquad\qquad c_2 = -1/2$

$y(t) = \frac{1}{2}\cos 2t - 1/2 \sin 2t$

$R(\text{amplitude}) = \sqrt{1/2^2 + (1/2)^2} = \sqrt{1/4 + 1/4} = \sqrt{1/2} = \frac{1}{\sqrt{2}} \approx .7071$

10. $k = 18,\ m = 2,\ \gamma = 4 \qquad y(0) = 1,\ y'(0) = 0$

$\frac{2y'' + 4y' + 18y = 0}{2} \Rightarrow y'' + 2y' + 9y = 0 \qquad r = \frac{-2 \pm \sqrt{4 - 36}}{2} =$

$r^2 + 2r + 9 = 0$

$\frac{-2 \pm \sqrt{-32}}{2} = \frac{-2 \pm 4\sqrt{2}i}{2}$

$y(t) = c_1 e^{-t}\cos(2\sqrt{2}t) + c_2 e^{-t}\sin(2\sqrt{2}t) \qquad = -1 \pm 2\sqrt{2}i$

$1 = c_1(1)(1) + \cancel{c_2(1)(0)} \qquad c_1 = 1$

$y'(t) = -\cancel{c_1} e^{-t}\cos(2\sqrt{2}t) - e^{-t}\sin(2\sqrt{2}t)2\sqrt{2} + c_2(-1)e^{-t}\sin(2\sqrt{2}t) + c_2 2\sqrt{2}e^{-t}\cos(2\sqrt{2}t)$

$0 = -(1)(1) - \cancel{(1)(0)}2\sqrt{2} + \cancel{c_2(-1)(0)} + 2\sqrt{2}c_2(1)(1)$

$1 = 2\sqrt{2}c_2 \Rightarrow \frac{1}{2\sqrt{2}}c_2 \Rightarrow \frac{\sqrt{2}}{4} = c_2$

$y(t) = e^{-t}\cos(2\sqrt{2}t) + \frac{\sqrt{2}}{4}e^{-t}\sin(2\sqrt{2}t) \qquad R = \sqrt{1^2 + (\frac{\sqrt{2}}{4})^2} = \sqrt{1 + \frac{2}{16}} = \sqrt{1 + \frac{1}{8}} = \sqrt{\frac{9}{8}}$

$\tan^{-1}(\frac{\sqrt{2}}{4}) \approx .3398 \qquad = \frac{3}{2\sqrt{2}}$

19.47° $\quad y(t) = \frac{3}{2\sqrt{2}}\cos(2\sqrt{2}t - .3398)$ System is underdamped

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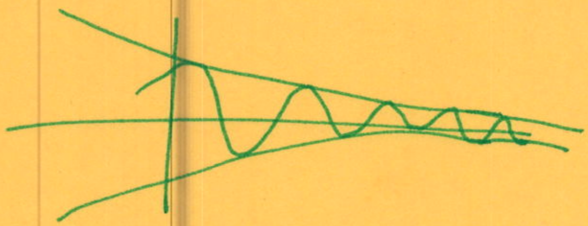
10 cont'd



11. $\frac{16}{32} = \frac{1}{2}$ slugs $= m$, $\gamma = 8$, $k = 7$

$\frac{1}{2}y'' + 8y' + 7y = 0$ or $y'' + 16y' + 14y = 0$

12. $y'' + 2y' + 10y = 4$ $\qquad r^2 + 2r + 10 = 0$ $\qquad y'' + 2y' + 10y = 4\cos 2t$

$r = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$

$y(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t$

$f(t) = 4$ $\qquad$ assume $Y(t) = A$ $\quad Y'(t) = 0$ $\quad Y''(t) = 0$

$0 + 2(0) + 10(A) = 4 \Rightarrow A = 4/10$

$y(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t + 4/10$

$g(t) = 4\cos 2t$

$Y(t) = A\cos 2t + B\sin 2t$ $\qquad Y'(t) = -2A\sin 2t + 2B\cos 2t$

$Y''(t) = -4A\cos 2t - 4B\sin 2t$

$$-4A\cos 2t - 4B\sin 2t + 2(-2A\sin 2t + 2B\cos 2t) + 10(A\cos 2t + B\sin 2t) = 4\cos 2t$$

$-4A + 4B + 10A = 4 \Rightarrow 6A + 4B = 4$

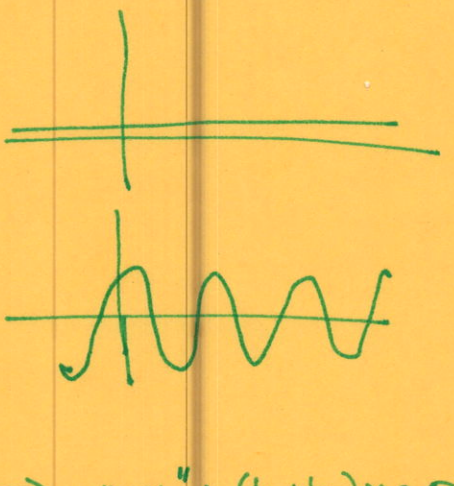
$-4B - 4A + 10B = 0 \qquad -4A + 6B = 0 \Rightarrow A = 6/13$

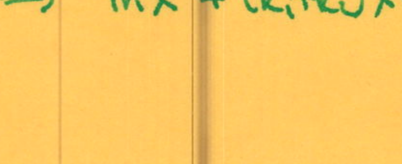
$B = 4/13$

$y(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t + 6/13\cos 2t + 4/13\sin 2t$

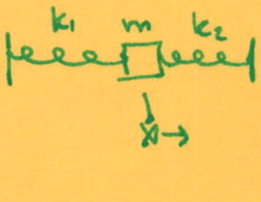
$f(t) = 4$

$g(t) = 4\cos 2t$





neither system experiences resonance since neither forcing function matches the frequency of the original unforced system

13. 

$mx'' = -k_1x - k_2x \Rightarrow mx'' + (k_1+k_2)x = 0$

212 Homework #6 key Cont'd

14. a. underdamped

b. undamped

c. critically damped or overdamped

15. a. $y'' + 4y = \cos(5/2\, t)$ beats or $y'' + 169y = \sin(12t)$

b. resonance $y'' + 4y = \cos 2t + \sin 2t$

$\boxed{\text{answers will vary}}$

c. $y'' + 6y' + 13y = 0 \rightarrow 0$

d. contains a transient solution

$y'' + 6y' + 13y = 0$ (or similar function w/ forcing)

e. oscillating

→ can be undamped $y'' + 4y = 0$ or forced transient w/ sin/cos fun

$y'' + 6y' + 13y = \sin t$

f. no damping $y' + 13y = 0$

g. critical damping ⇒ repeated roots

$y'' + 4y' + 4y = 0$

$9y'' + 12y' + 16y = 0$