212 Homework #7 Key

1a. $1^{1/3}$

$$1 = e^{0\pi i} = e^{2\pi i} = e^{4\pi i}$$

$$1^{1/3} = e^{0\pi/3 i}, e^{2\pi/3 i}, e^{4\pi/3 i}$$

$$e^{0i} = 1$$

$$e^{2\pi/3 i} = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + \sqrt{3}/2 i$$

$$e^{4\pi/3 i} = \cos(4\pi/3) + i\sin(4\pi/3)$$

$$-1/2 - \sqrt{3}/2 i$$

b. $(1-i)^{1/2}$ $\quad ||1-i|| = \sqrt{1^2+1^2} = \sqrt{2}$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - 1/\sqrt{2} i\right) = \sqrt{2} e^{3\pi/4 i} \Rightarrow \sqrt{1-i} = \left(\sqrt{2} e^{3\pi/4 i}\right)^{1/2} = \left(\sqrt{2} e^{11\pi/4 i}\right)^{1/2}$$

$$\sqrt[4]{2} e^{3\pi/8 i}, \sqrt[4]{2} e^{11\pi/8 i}$$

$$\sqrt[4]{2}\left[\cos(3\pi/8) + i\sin(3\pi/8)\right] = \sqrt[4]{2}\cos(3\pi/8) + i\sqrt[4]{2}\sin(3\pi/8)$$

$$\sqrt[4]{2}\left[\cos(11\pi/8) + i\sin(11\pi/8)\right] = \sqrt[4]{2}\cos(11\pi/8) + i\sqrt[4]{2}\sin(11\pi/8)$$

c. $(-4i)^{1/4} = \left(4e^{3\pi/2 i}\right)^{1/4}$ $\quad (4)^{1/4} = 2^{1/2} = \sqrt{2}$

$$e^{7\pi/2 i}$$

$$e^{11\pi/2 i}$$

$$e^{15\pi/2 i}$$

$$\sqrt{2} e^{3\pi/8 i} = \sqrt{2}\left[\cos(3\pi/8) + i\sin(3\pi/8)\right] = \sqrt{2}\cos(3\pi/8) + i\sqrt{2}\sin(3\pi/8)$$

$$\sqrt{2} e^{7\pi/8 i} = \sqrt{2}\left[\cos(7\pi/2) + i\sin(7\pi/2)\right] = \sqrt{2}\cos(7\pi/2) + i\sqrt{2}\sin(7\pi/8)$$

$$\sqrt{2} e^{11\pi/8 i} = \sqrt{2}\left[\cos(11\pi/8) + i\sin(11\pi/8)\right] = \sqrt{2}\cos(11\pi/8) + i\sqrt{2}\sin(11\pi/8)$$

$$\sqrt{2} e^{15\pi/8 i} = \sqrt{2}\left[\cos(15\pi/8) + i\sin(15\pi/8)\right] = \sqrt{2}\cos(15\pi/8) + i\sqrt{2}\sin(15\pi/8)$$

d. $(1-\sqrt{3}i)^{1/6}$ $\quad ||1-\sqrt{3}|| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

$$\left[\frac{1}{2}\left(1/2 - \frac{\sqrt{3}}{2}i\right)\right]^{1/6} = \left[\frac{1}{2} e^{11\pi/6 i}\right]^{1/6} \qquad \left(\frac{1}{2}\right)^{1/6} = \frac{1}{\sqrt[6]{2}}$$

$$e^{11\pi/6 i} = e^{23\pi/6 i} = e^{35\pi/6 i} = e^{47\pi/6 i} = e^{59\pi/6 i} = e^{71\pi/6 i}$$

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1d. cont'd

$\frac{1}{\sqrt[6]{2}}\left[\cos\left(11\pi/36\right) + i\sin\left(11\pi/36\right)\right] = \frac{1}{\sqrt[6]{2}}\cos\left(11\pi/36\right) + i\frac{1}{\sqrt[6]{2}}\sin\left(11\pi/36\right)$,

$\frac{1}{\sqrt[6]{2}}\cos\left(23\pi/36\right) + i\frac{1}{\sqrt[6]{2}}\sin\left(\frac{23\pi}{36}\right)$, $\frac{1}{\sqrt[6]{2}}\cos\left(\frac{35\pi}{36}\right) + i\frac{1}{\sqrt[6]{2}}\sin\left(35\pi/36\right)$,

$\frac{1}{\sqrt[6]{2}}\cos\left(47\pi/36\right) + i\frac{1}{\sqrt[6]{2}}\sin\left(47\pi/36\right)$, $\frac{1}{\sqrt[6]{2}}\cos\left(\frac{59\pi}{36}\right) + i\frac{1}{\sqrt[6]{2}}\sin\left(59\pi/36\right)$,

$\frac{1}{\sqrt[6]{2}}\cos\left(71\pi/36\right) + i\frac{1}{\sqrt[6]{2}}\sin\left(71\pi/36\right)$.

e. $(-1)^{\frac{1}{5}} = \left(e^{\pi i}\right)^{1/5}\ \left(e^{3\pi i}\right)^{1/5}\ \left(e^{5\pi}\right)^{1/5}, \left(e^{7\pi}\right)^{1/5}, \left(e^{9\pi}\right)^{1/5}$

$\cos(\pi/5) + i\sin(\pi/5)$, $\cos(3\pi/5) + i\sin(3\pi/5)$, $\cos(\pi) + i\sin\pi$
$= -1$,

$\cos(7\pi/5) + i\sin(7\pi/5)$, $\cos(9\pi/5) + i\sin(9\pi/5)$

2a. $y^{iv} + 4y''' + 3y = t$ $p(t) = 4$ defined everywhere

b. $y''' + ty'' + t^2y' + t^3y = \ln t$ $p(t) = t$ defined everywhere $t \geq 0$

3a. $y_1 = 1$ $y_1' = 0$ $y_1'' = 0$ $y_1''' = 0$ $y''' + y' = 0$ $0 + 0 = 0$ ✓

$y_2 = \cos t$ $y_2' = -\sin t$ $y_2'' = -\cos t$, $y_2''' = \sin t$ $-\sin t + \sin t = 0$ ✓

$y_3 = \sin t$ $y_3' = \cos t$ $y_3'' = -\sin t$ $y_3''' = -\cos t$ $\cos t - \cos t = 0$ ✓

$\begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = (\sin^2 t + \cos^2 t)1 = (1)(1) = 1$ ✓ yes, fundamental set

b. $y^{iv} + 2y''' + y'' = 0$ $0 + 2(0) + 0 = 0$ ✓

$y_1 = 1$ $y_1' = 0$ $y_1'' = 0$ $y_1''' = 0$, $y_1^{iv} = 0$ $0 + 2(0) + 0 = 0$ ✓

$y_2 = t$ $y_2' = 1$ $y_2'' = 0$ $y_2''' = 0$ $y_2^{iv} = 0$ $e^{-t} + 2(-e^{-t}) + e^{-t} = 0$ ✓

$y_3 = e^{-t}$ $y_3' = -e^{-t}$ $y_3'' = e^{-t}$ $y_3''' = -e^{-t}$, $y_3^{iv} = e^{-t}$

$y_4 = te^{-t}$ $y_4' = e^{-t} - te^{-t}$, $y_4'' = (-1)e^{-t} + (1-t)(-e^{-t})$ $y_4''' = e^{-t} - (t-2)e^{-t}$ $y_4^{iv} =$
$(1-t)e^{-t}$ $-(2-t)e^{-t}$ $-1+t$ $(3-t)e^{-t}$ $-e^{-t} - (3-t)e^{-t}$
$(t-2)$ $(t-4)e^{-4}$

36. cont'd

$$(t-4)e^{-4}+2\left[(3-t)e^{-t}\right]+(t-2)e^{-t}=e^{-t}\left[\cancel{t}-\cancel{4}+\cancel{6}-\cancel{2t}+\cancel{t}-\cancel{2}\right]=0 \checkmark$$

$$\begin{vmatrix} 1 & t & e^{-t} & te^{-t} \\ 0 & 1 & -e^{-t} & (t-1)e^{-t} \\ 0 & 0 & e^{-t} & (2-t)e^{-t} \\ 0 & 0 & -e^{-t} & (t-3)e^{-t} \end{vmatrix} = 1\begin{vmatrix} 1 & -e^{-t} & (t-1)e^{-t} \\ 0 & e^{-t} & (2-t)e^{-t} \\ 0 & -e^{-t} & (t-3)e^{-t} \end{vmatrix} = (1)(1)\begin{vmatrix} e^{-t} & (2-t)e^{-t} \\ -e^{-t} & (t-3)e^{-t} \end{vmatrix} =$$

$$e^{-2t}(t-3)+e^{-2t}(2-t)=e^{-2t}\left[\cancel{t}-3+2-\cancel{t}\right]=-e^{-2t}\neq 0$$

yes, it is a fundamental set

c. $x^3y'''+x^2y''-2xy'+2y=0 \quad x, x^2, \frac{1}{x}$

$y_1=x \quad y_1'=1 \quad y_1''=0 \quad y_1'''=0 \quad x^3(0)+x^2(0)-2x(1)+2(x)=0 \checkmark$

$y_2=x^2 \quad y_2'=2x \quad y_2''=2 \quad y_2'''=0 \quad x^3(0)+x^2(2)-2x(2x)+2(x^2)=0 \checkmark$

$y_3=x^{-1} \quad y_3'=-x^{-2} \quad y_3''=2x^{-3} \quad y_3'''=-6x^{-4} \quad x^3\left(\frac{-6}{x^4}\right)+x^2\left(\frac{2}{x^3}\right)-2x\left(\frac{-1}{x^2}\right)+2\left(\frac{1}{x}\right)$

$\frac{1}{x} \quad -\frac{1}{x^2} \quad \frac{2}{x^3} \quad \frac{-6}{x^4}$

$-\frac{6}{x}+\frac{2}{x}+\frac{2}{x}+\frac{2}{x}=0 \checkmark$

$$\begin{vmatrix} x & x^2 & 1/x \\ 1 & 2x & -1/x^2 \\ 0 & 2 & 2/x^3 \end{vmatrix} = x\begin{vmatrix} 2x & -1/x^2 \\ 2 & 2/x^3 \end{vmatrix} - 1\begin{vmatrix} x^2 & 1/x \\ 2 & 2/x^3 \end{vmatrix} + 0\cancel{\begin{vmatrix} x^2 & 1/x \\ 2x & -1/x^2 \end{vmatrix}}$$

$$= x\left(\frac{4x}{x^3}+\frac{2}{x^2}\right)-1\left(\frac{2x^2}{x^3}-\frac{2}{x}\right)=\cancel{x}\left(\frac{6}{\cancel{x}}\right)-1(0)=6\neq 0$$

yes, this is a fundamental set

d. $y^{iv}-y=0 \quad y(t)=c_1\cos t+c_2\sin t+c_3\cosh t+c_4\sinh t$

$y'=-c_1\sin t+c_2\cos t+c_3\sinh t+c_4\cosh t$

$y''=-c_1\cos t-c_2\sin t+c_3\cosh t+c_4\sinh t$

$y'''=c_1\sin t-c_2\cos t+c_3\sinh t+c_4\cosh t$

$y^{iv}=c_1\cos t+c_2\sin t+c_3\cosh t+c_4\sinh t$

$c_1\cos t+c_2\sin t+c_3\cosh t+c_4\sinh t-c_1\cos t-c_2\sin t-c_3\cosh t-c_4\sinh t=0 \checkmark$

$$\begin{vmatrix} \cos t & \sin t & \cosh t & \sinh t \\ -\sin t & \cos t & \sinh t & \cosh t \\ -\cos t & -\sin t & \cosh t & \sinh t \\ \sin t & -\cos t & \sinh t & \cosh t \end{vmatrix} = \cos t\begin{vmatrix} \cos t & \sinh t & \cosh t \\ -\sin t & \cosh t & \sinh t \\ -\cos t & \sinh t & \cosh t \end{vmatrix} - \sin t\begin{vmatrix} -\sin t & \sinh t & \cosh t \\ -\cos t & \cosh t & \sinh t \\ \sin t & \sinh t & \cosh t \end{vmatrix}$$

$$+\cosh t\begin{vmatrix} -\sin t & \cos t & \cosh t \\ -\cos t & -\sin t & \sinh t \\ \sin t & -\cos t & \cosh t \end{vmatrix} - \sinh t\begin{vmatrix} -\sin t & \cos t & \sinh t \\ -\cos t & -\sin t & \cosh t \\ \sin t & -\cos t & \sinh t \end{vmatrix}$$

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3d cont'd

$$\begin{vmatrix} \cos t & \sinh t & \cosh t \\ -\sin t & \cosh t & \sinh t \\ -\cos t & \sinh t & \cosh t \end{vmatrix} = \cos t \begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} + \sin t \begin{vmatrix} \sinh t & \cosh t \\ \sinh t & \cosh t \end{vmatrix} - \cos t \begin{vmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{vmatrix}$$

$\cosh^2 t - \sinh^2 t = 1$ $\quad$ $\sinh t \cosh t - \sinh t \cosh t = 0$ $\quad$ $\sinh^2 t - \cosh^2 t = -1$

$$\cos t(1) + 0 - \cos t(-1) = 2\cos t$$

$$\begin{vmatrix} -\sin t & \sinh t & \cosh t \\ -\cos t & \cosh t & \sinh t \\ \sin t & \sinh t & \cosh t \end{vmatrix} = -\sin t \begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} + \cos t \begin{vmatrix} \sinh t & \cosh t \\ \sinh t & \cosh t \end{vmatrix} + \sin t \begin{vmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{vmatrix}$$

$\cosh^2 t - \sinh^2 t = 1$ $\quad$ $\sinh t \cosh t - \sinh t \cosh t = 0$ $\quad$ $\sinh^2 t - \cosh^2 t = -1$

$$-\sin t(1) + 0 + \sin(t)(-1) = -2\sin t$$

$$\begin{vmatrix} -\sin t & \cos t & \cosh t \\ -\cos t & -\sin t & \sinh t \\ \sin t & -\cos t & \cosh t \end{vmatrix} = \cosh t \begin{vmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{vmatrix} - \sinh t \begin{vmatrix} -\sin t & \cos t \\ \sin t & -\cos t \end{vmatrix} + \cosh t \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix}$$

$\cos^2 t + \sin^2 t = 1$ $\quad$ $\sin t \cos t - \sin t \cos t = 0$ $\quad$ $\cos^2 t + \sin^2 t = 1$

$$\cosh t(1) \underset{+0}{+} \cosh(t)(1) = 2\cosh t$$

$$\begin{vmatrix} -\sin t & \cos t & \sinh t \\ -\cos t & -\sin t & \cosh t \\ \sin t & -\cos t & \sinh t \end{vmatrix} = \sinh t \begin{vmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{vmatrix} - \cosh t \begin{vmatrix} -\sin t & \cos t \\ \sin t & -\cos t \end{vmatrix} + \sinh t \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix}$$

$\cos^2 t + \sin^2 t = 1$ $\quad$ $\cos t \sin t - \cos t \sin t = 0$ $\quad$ $\sin^2 t + \cos^2 t = 1$

$$\sinh t(1) + 0 + \sinh t(1) = 2\sinh t$$

$$\cos t(2\cos t) - \sin t(-2\sin t) + \cosh t(2\cosh t) - \sinh t(2\sinh t)$$

$$= 2\cos^2 t + 2\sin^2 t + 2\cosh^2 t - 2\sinh^2 t$$

$$= 2\underbrace{(\cos^2 t + \sin^2 t)}_{=1} + 2(\underbrace{\cosh^2 t - \sinh^2 t}_{=1})$$

$$2(1) + 2(1) = 4 \neq 0$$ yes, This is a fundamental set

4. $(2-t)y''' + (2t-3)y'' - ty' + y = 0 \quad t<2 \quad Y_1(t) = e^t \qquad Y_2(t) = vY_1 = ve^t$

$Y_2' = v'e^t + ve^t \quad Y_2'' = v''e^t + 2v'e^t + ve^t \quad Y_2''' = v'''e^t + 3v''e^t + 3v'e^t + ve^t$

$2v'''e^t + 6v''e^t + 6v'e^t + 2ve^t - tv'''e^t - 3tv''e^t - 3tv'e^t - tve^t + 2tv''e^t + 4tv'e^t + 2tve^t$

$-3v''e^t - 6v'e^t - 3ve^t - tv'e^t - tve^t + ve^t = 0 \qquad (\div e^t)$; collect v

$2v$[illegible] $-tv + 2tv$[illegible] $-3v - tv + v = 0$

4 cont'd.

collect v'

$6v' - 3tv' + 4tv' - 6v' - tv' = 0$

collect v''

$6v'' - 3tv'' + 2tv'' - 3v'' = (3-t)v''$

collect v'''

$2v''' - tv'''$

$$\Rightarrow (3-t)v'' + (2-t)v''' = 0$$

let $u = v''$

$\frac{du}{dt} = v'''$

$$(3-t)u + (2-t)\frac{du}{dt} = 0$$

$$(2-t)\frac{du}{dt} = (t-3)u$$

$$\int \frac{du}{du} = \int \frac{-t+3}{t-2}\,dt$$

$$\begin{array}{r|l} & -1 \\ \hline t-2 & -t+3 \\ & +t \mp 2 \\ \hline & 1 \end{array}$$

$\int \frac{du}{u} = \int -1 + \frac{1}{t-2}\,dt \Rightarrow \ln u = -t + \ln|t-2| + C$

$C = \ln e^{A}$

$\ln u = \ln(e^{-t}) + \ln(t-2) + \ln e^{A} = \ln\left[Ae^{-t}(t-2)\right]$

$u = Ae^{-t}(t-2) = v''$

	u	dv
+	t-2	e^{-t}
−	1	$-e^{-t}$
+	0	e^{-t}

$v' = \int e^{-t}(t-2)\,dt$

$= -(t-2)e^{-t} - e^{-t} =$

$2e^{-t} \quad te^{-t} - e^{-t} = (1-t)e^{-t} + C$

$V = \int (1-t)e^{-t}\,dt$

	u	dv
+	1-t	e^{-t}
−	-1	$-e^{-t}$
+	0	e^{-t}

$= -(1-t)e^{-t} + e^{-t} + ct + D$

$\cancel{-e^{-t}} + \cancel{e^{-t}} + te^{-t} + ct + D$

$V = te^{-t} + ct + D$

$Y_2 = e^{t}(te^{-t} + ct + D) = t + Cte^{t} + \underbrace{De^{t}}_{y_1}$

$Y_2 = t + Cte^{t}$ or better

$Y_2 = t, \ Y_3 = te^{t} \qquad Y_1 = e^{t}$

3rd order, 3 solutions

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5c. cont'd

$$y(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + c_3 e^{-2t}\cos\sqrt{3}t + c_4 e^{-2t}\sin\sqrt{3}t$$

$$y'(t) = -c_1 e^{-t}\cos t - c_1 e^{-t}\sin t - c_2 e^{-t}\sin t + c_2 e^{-t}\cos t - 2c_3 e^{-2t}\cos\sqrt{3}t - \sqrt{3}c_3 e^{-2t}\sin\sqrt{3}t$$
$$-2c_4 e^{-2t}\sin\sqrt{3}t + \sqrt{3}c_4 e^{-2t}\cos\sqrt{3}t$$

$$y''(t) = c_1\left[\cancel{e^{-t}\cos t} - 2e^{-t}\sin t - \cancel{e^{-t}\cos t}\right] + c_2\left[\cancel{e^{-t}\sin t} + 2e^{-t}\cos t - \cancel{e^{-t}\sin t}\right]$$
$$+ c_3\left[4e^{-2t}\cos\sqrt{3}t + 2(-2)e^{-2t}(-\sqrt{3}\sin\sqrt{3}t) + e^{-2t}\cdot(-3\cos\sqrt{3}t)\right]$$
$$+ c_4\left[4e^{-2t}\sin\sqrt{3}t + 2(-2)e^{-2t}\sqrt{3}\cos\sqrt{3}t + {}^{-}e^{-2t}3\sin\sqrt{3}t\right]$$

$$y'''(t) = c_1\left[2e^{-t}\sin t - 2e^{-t}\cos t\right] + c_2\left[-2e^{-t}\cos t - 2e^{-t}\sin t\right] +$$
$$c_3\left[-8e^{-2t}\cos\sqrt{3}t + 3\cdot 4e^{-2t}(-\sqrt{3}\sin\sqrt{3}t) + 3(-2e^{-2t})(-3\cos\sqrt{3}t) + e^{-2t}\cdot 3\sqrt{3}\sin\sqrt{3}t\right]$$
$$+ c_4\left[-8e^{-2t}\sin\sqrt{3}t + 3\cdot 4e^{-2t}\sqrt{3}\cos\sqrt{3}t + 3\cdot(-2e^{-2t})(-3\sin\sqrt{3}t) + e^{-2t}(-3\sqrt{3}\cos\sqrt{3}t)\right]$$

$y(0)$: $c_1(1)(1) + \cancel{c_2(1)(0)}^{0} + c_3(1)(1) + \cancel{c_4(1)(0)}$ $\qquad c_1 + c_3 = 1$

$y'(0)$: $-c_1(1)(1) - \cancel{c_1(1)(0)}^{0} - \cancel{c_2(1)(0)}^{0} + c_2(1)(1) - 2c_3(1)(1) - \sqrt{3}\,\cancel{c_3(1)}(0) -$
$\cancel{2c_4(1)(0)}^{0} + \sqrt{3}c_4(1)(1)$

$$-c_1 + c_2 - 2c_3 + \sqrt{3}c_4 = -2$$

$y''(0)$: $c_1\cancel{(-2)(1)}^{0}(0) + c_2(2)(1)(1) + c_3\left[4(1)(1) + 2\cancel{(-2)(1)(-\sqrt{3})}(0) + (1)(-3)(1)\right]$
$+ c_4\left[4\cancel{(1)(0)}^{0} + 2(-2)(1)(\sqrt{3})(1) + \cancel{(-1)(1)(3)}(0)\right]$

$$2c_2 + c_3 - 4\sqrt{3}c_4 = 0$$

$y'''(0)$: $c_1\left[\cancel{2(1)(0)}^{0} - 2(1)(1)\right] + c_2\left[(-2)(1)(1) - \cancel{2(1)(0)}\right] + c_3\left[(-8)(1)(1) + \cancel{(3)(4)(1)(-\sqrt{3})(0)}^{0}\right.$
$+ (3)(-2)(1)(-3)(1) + \cancel{(1)(3\sqrt{3})(0)}^{0}\Big] + c_4\Big[\cancel{-8(1)(0)}^{0} + (3)(4)(1)(\sqrt{3})(1) + \cancel{3(-2)(1)(-3)(0)}^{0}$
$\left. + (1)(-3\sqrt{3})(1)\right]$

$$-2c_1 - 2c_2 + 10c_3 + 9\sqrt{3}c_4 = 3$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ -1 & 1 & -2 & \sqrt{3} & -2 \\ 0 & 2 & 1 & -4\sqrt{3} & 0 \\ -2 & -2 & 10 & 9\sqrt{3} & 3 \end{array}\right] \Rightarrow \begin{bmatrix} -3/13 \\ 14/13 \\ 16/13 \\ -11/13\sqrt{3} \end{bmatrix}$$

$$y(t) = -\tfrac{3}{13}e^{-t}\cos t + \tfrac{14}{13}e^{-t}\sin t + \tfrac{16}{13}e^{-2t}\cos\sqrt{3}t - \tfrac{11}{13\sqrt{3}}e^{-2t}\sin\sqrt{3}t$$

5a. $y''' + y' = 0 \qquad y(0) = 0,\ y'(0) = 1,\ y''(0) = 2$

$r^3 + r = 0$

$r(r^2+1) = 0 \Rightarrow r = 0,\ r = \pm i \qquad e^0 = 1$

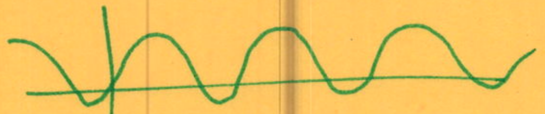
$y(t) = c_1 + c_2\cos t + c_3\sin t \qquad c_1 + c_2 = 0$

$y'(t) = -c_2\sin t + c_3\cos t \qquad c_3 = 1$

$y''(t) = -c_2\cos t - c_3\sin t \qquad -c_2 = 2 \Rightarrow c_2 = -2$

$c_1 = 2$

$y(t) = 2 - 2\cos t + \sin t$



b. $y^{iv} - 4y''' + 4y'' = 0 \qquad y(1) = -1,\ y'(1) = 2,\ y''(1) = 0,\ y'''(1) = 0$

$r^4 - 4r^3 + 4r^2 = 0$

$r^2(r^2 - 4r + 4) = 0 \qquad r^2 = 0 \qquad e^0 = 1,\ te^0 = t$

$(r-2)^2 = 0 \qquad r = 2 \qquad$ repeated $e^{2t},\ te^{2t}$

$y(t) = c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t} \qquad c_1 + c_2 + c_3 e^2 + c_4 e^2 = -1$

$y'(t) = c_2 + 2c_3 e^{2t} + c_4 e^{2t} + 2c_4 t e^{2t} \qquad c_2 + 2c_3 e^2 + c_4 e^2 + 2c_4 e^2 = 2$

$y''(t) = 4c_3 e^{2t} + \underbrace{2c_4 e^{2t} + 2c_4 e^{2t}} + 4c_4 t e^{2t} \qquad 4c_3 e^2 + 2c_4 e^2 + 2c_4 e^2 + 4c_4 e^2 = 0$

$y'''(t) = 8c_3 e^{2t} + 8c_4 e^{2t} + 4c_4 e^{2t} + 8c_4 t e^{2t} \qquad 8c_3 e^2 + 12c_4 e^2 + 8c_4 e^2 = 0$

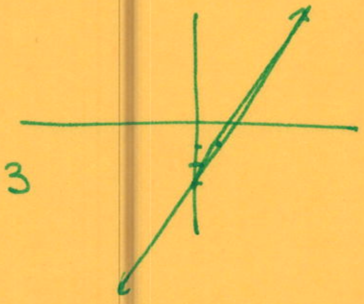
÷ last 2 eq. by e^2 $\qquad 4c_3 + 8c_4 = 0 \qquad \Rightarrow c_3 = c_4 = 0$

$8c_3 + 20c_4 = 0$

reduces top 2 equations to $\quad c_1 + c_2 = -1 \Rightarrow c_1 + 2 = -1$

$c_2 = 2 \qquad c_1 = -3$

$y(t) = -3 + 2t$



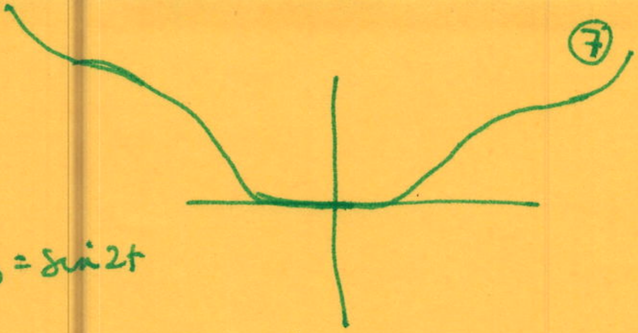
c. $y^{iv} + 6y''' + 17y'' + 22y' + 14y = 0$

$y(0) = 1,\ y'(0) = -2,\ y''(0) = 0,\ y'''(0) = 3$

$r^4 + 6r^3 + 17r^2 + 22r + 14 = 0$

$(r^2 + 2r + 2)(r^2 + 4r + 7) = 0$

$\dfrac{-2 \pm \sqrt{4-8}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i \qquad \dfrac{-4 \pm \sqrt{16-28}}{2} = \dfrac{-4 \pm \sqrt{12}\,i}{2} = \dfrac{-4 \pm 2\sqrt{3}i}{2} = -2 \pm \sqrt{3}i$



6a. $y''' + 4y' = t \quad y(0) = y'(0) = 0, \; y''(0) = 1$

$r^3 + 4r = 0 \qquad r = 0 \Rightarrow y_1 = 1$

$r(r^2+4) = 0 \qquad r = \pm 2i \quad y_2 = \cos 2t, \; y_3 = \sin 2t$

$t(At+B) = Y(t) = At^2 + Bt \qquad 0 + 4(2At) = t$

$\uparrow \qquad Y'(t) = 2At + B \qquad 8A = 1$

mult. since 1 is repeated (1 is a solution) $\quad Y''(t) = 2A \qquad Y'''(t) = 0 \qquad A = \frac{1}{8}$

$y(t) = c_1 + c_2 \cos 2t + c_3 \sin 2t + \frac{1}{8}t^2$

$c_1 + c_2(1) + c_3(0) + \frac{1}{8}(0) = 0 \Rightarrow c_1 + c_2 = 0$

$y'(t) = -2c_2 \sin 2t + 2c_3 \cos 2t + \frac{1}{4}t$

$-2c_2(0) + 2c_3(1) + \frac{1}{4}(0) = 0 \qquad 2c_3 = 0$
$c_3 = 0$

$y''(t) = -4c_2 \cos 2t + \frac{1}{4}$

$-4c_2(1) + \frac{1}{4} = 1 \Rightarrow -4c_2 = 3/4 \Rightarrow c_2 = -\frac{3}{16}$

$c_1 = \frac{3}{16}$

$y(t) = \frac{3}{16} - \frac{3}{16}\cos 2t + \frac{1}{8}t^2$

6b. $y^{iv} + 2y'' + y = 3t+4 \quad y(0) = y'(0) = 0 \quad y''(0) = y'''(0) = 1$

$r^4 + 2r^2 + 1 = 0$

$(r^2+1)^2 = 0 \qquad r = \pm i$ repeated

$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$

$Y(t) = At + B \qquad Y'(t) = A, \; Y''(t) = 0 \quad Y'''(t) = 0 \quad Y^{iv}(t) = 0$

$At + B = 3t + 4 \Rightarrow A = 3, B = 4$

$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t + 3t + 4$

$y'(t) = -c_1 \sin t + c_2 \cos t + c_3[\cos t + t(-\sin t)] + c_4[\sin t + t\cos t] + 3$

$y''(t) = -c_1 \cos t - c_2 \sin t + c_3[0 \cdot \cos t + 2(1)(-\sin t) - t(\cos t)] + c_4[0 \cdot \sin t + 2(1)\cos t + t(-\sin t)] + 0$

$y'''(t) = c_1 \sin t - c_2 \cos t + c_3[0 \cdot \cos t + 3(0)(-\sin t) + 3(1)(-\cos t) + t\sin t] +$
$c_4[0 \cdot \sin t + 3(0)(\cos t) + 3(1)(-\sin t) + t(-\cos t)]$

$y(0)$: $c_1(1) + c_2(0) + c_3(0)(1) + c_4(0)(0) + 3(0) + 4 = 0 \qquad c_1 + 4 = 0 \Rightarrow c_1 = -4$

$y'(0)$: $-c_1(0) + c_2(1) + c_3[(1) + (0)(0)] + c_4[(0) + (0)(1)] + 3 = 0 \quad c_2 + c_3 + 3 = 0$

$y''(0)$: $-c_1(1) - c_2(0) + c_3[(0)(1) + (2)(0) - (0)(1)] + c_4[0(0) + 2(1) + (0)(0)] = 1$

$-c_1 + 2c_4 = 1$

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6b cont'd.

$$y'''(0): c_1(0) - c_2(1) + c_3[(0)(1) + 3(0)(0) + 3(1)(-1) + 0(0)] + c_4[0(0) + 3(0)(1) + 3(1)(0) + (0)(-1)] = 1$$

$$-c_2 - 3c_3 = 1$$

$c_1 = -4$

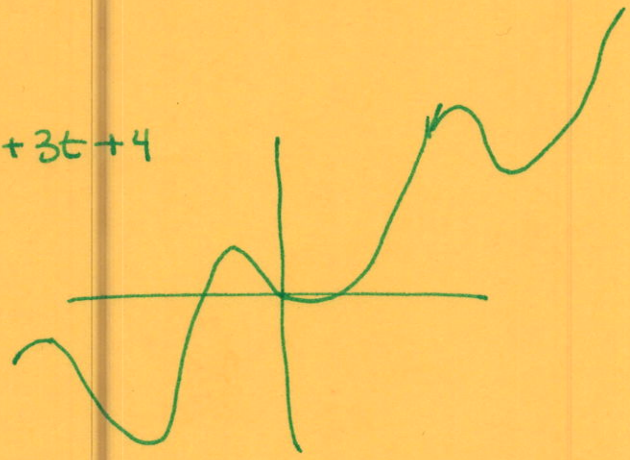
$-(-4) + 2c_4 = 1 \Rightarrow 4 + 2c_4 = 1 \Rightarrow 2c_4 = -3 \Rightarrow c_4 = -3/2$

$$\begin{array}{l} c_2 + c_3 = -3 \\ \underline{-c_2 - 3c_3 = 1} \\ \quad -2c_3 = -2 \Rightarrow c_3 = 1 \end{array}$$

$c_2 + 1 = -3$

$c_2 = -4$

$$y(t) = -4\cos t - 4\sin t + t\cos t - 3/2\, t\sin t + 3t + 4$$



7. $y''' - y'' + y' - y = \sec t$

$y(0) = 2,\ y'(0) = 1,\ y''(0) = 1$

$r^3 - r^2 + r - 1 = 0$

$r^2(r-1) + 1(r-1) = 0 \Rightarrow (r-1)(r^2+1) = 0$

$r = 1,\ r = \pm i \qquad y_1 = e^t,\ y_2 = \cos t,\ y_3 = \sin t$

$$W = \begin{vmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{vmatrix} = e^t \underbrace{\begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix}}_{=1} - e^t \underbrace{\begin{vmatrix} \cos t & \sin t \\ -\cos t & -\sin t \end{vmatrix}}_{=0} + e^t \underbrace{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}}_{=1} = 2e^t$$

$$W_1 = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1 \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1[\cos^2 t + \sin^2 t] = 1$$

$$W_2 = \begin{vmatrix} e^t & 0 & \sin t \\ e^t & 0 & \cos t \\ e^t & 1 & -\sin t \end{vmatrix} = 1 \begin{vmatrix} e^t & \sin t \\ e^t & \cos t \end{vmatrix} = e^t\cos t - e^t\sin t$$

$$W_3 = \begin{vmatrix} e^t & \cos t & 0 \\ e^t & -\sin t & 0 \\ e^t & -\cos t & 1 \end{vmatrix} = 1 \begin{vmatrix} e^t & \cos t \\ e^t & -\sin t \end{vmatrix} = -e^t\sin t - e^t\cos t$$

$$Y(t) = y_1 \int \frac{\sec t\,(1)}{2e^t}\,dt + y_2 \int \frac{\sec t\,(e^t\cos t - e^t\sin t)}{2e^t}\,dt + y_3 \int \frac{\sec t\,(-e^t\sin t - e^t\cos t)}{2e^t}\,dt$$

7 cont'd.

$$Y(t) = \frac{1}{2}e^t\int e^{-t}\sec t\,dt + \frac{1}{2}\cos t\int 1 - \tan t\,dt - \frac{1}{2}\sin t\int 1 + \tan t\,dt$$

$$= \frac{1}{2}e^t\int_0^t e^{-s}\sec s\,ds + \frac{1}{2}\cos t\left[t + \ln|\cos t|\right] - \frac{1}{2}\sin t\left[t - \ln|\cos t|\right]$$

$$y(t) = c_1e^t + c_2\cos t + c_3\sin t + \frac{1}{2}e^t\int_0^t e^{-s}\sec s\,ds + \frac{1}{2}t\cos t + \frac{1}{2}\cos t\ln|\cos t| -$$

$$\frac{1}{2}t\sin t + \frac{1}{2}\sin t\ln|\cos t|$$

$y(0)$: $c_1(1) + c_2(1) + \cancelto{0}{c_3(0)} + \frac{1}{2}(1)\int_0^1 e^{-s}\sec s\,ds + \cancelto{0}{\frac{1}{2}(0)(1)} + \cancelto{0}{\frac{1}{2}(1)\ln(1)} + \cancelto{0}{\frac{1}{2}(0)(0)} + \cancelto{0}{\frac{1}{2}(0)\ln(1)}$

$$c_1 + c_2 + \frac{1}{2}\int_0^1 e^{-s}\sec s\,ds = 2$$

$$y'(t) = c_1e^t - c_2\sin t + c_3\cos t + \frac{1}{2}e^t\int_0^t e^{-s}\sec s\,ds + \frac{1}{2}\cancel{e^t}(\cancel{e^{-t}}\sec t) + \frac{1}{2}\cos t$$

$$-\frac{1}{2}t\sin t - \frac{1}{2}\sin t\ln|\cos t| + \frac{1}{2}\underbrace{\cos t\cdot -\tan t}_{=\sin t} - \frac{1}{2}\sin t - \frac{1}{2}t\cos t$$

$$+ \frac{1}{2}\cos t\ln|\cos t| - \frac{1}{2}\sin t\cdot\tan t$$

$y'(0)$: $c_1(1) - \cancelto{0}{c_2(0)} + c_3(1) + \frac{1}{2}(1)\int_0^1 e^{-s}\sec s\,ds + \frac{1}{2}(1) + \frac{1}{2}(1) - \cancelto{0}{\frac{1}{2}(0)(0)} - \cancelto{0}{\frac{1}{2}(0)\ln(1)}$

$+ \cancelto{0}{\frac{1}{2}(1)(0)} - \cancel{\frac{1}{2}(0)} - \cancel{(1/2)(0)(1)}$

$$c_1 + c_3 + \frac{1}{2}\int_0^1 e^{-s}\sec s\,ds + 1 = 1$$

$$y''(t) = c_1e^t - c_2\cos t - c_3\sin t + \frac{1}{2}e^t\int_0^t e^{-s}\sec s\,ds + \frac{1}{2}\cancel{e^t}\cancel{e^{-t}}\sec t + \frac{1}{2}\sec t\tan t$$

$$+ -\frac{1}{2}\sin t - \frac{1}{2}\sin t - \frac{1}{2}t\cos t - \frac{1}{2}\cos t\ln|\cos t| + \frac{1}{2}\sin t\cdot\tan t + \frac{1}{2}\cos t$$

$$-\frac{1}{2}\cos t - \frac{1}{2}\cos t + \frac{1}{2}t\sin t + -\frac{1}{2}\sin t\ln|\cos t| + \frac{1}{2}\cos t(-\tan t)$$

$$-\frac{1}{2}\cos t\cdot\tan t - \frac{1}{2}\sin t\sec^2 t$$

$y''(0)$: $c_1(1) - c_2(1) - \cancelto{0}{c_3(0)} + \frac{1}{2}(1)\int_0^1 e^{-s}\sec s\,ds + \frac{1}{2}(1) + \cancelto{0}{\frac{1}{2}(1)(0)} - \cancelto{0}{\frac{1}{2}(0)} - \cancelto{0}{\frac{1}{2}(0)} - \cancelto{0}{\frac{1}{2}(0)(1)}$

$- \cancelto{0}{\frac{1}{2}(1)\ln(1)} + \cancelto{0}{\frac{1}{2}(0)(0)} + \frac{1}{2}(1) - \cancelto{0}{\frac{1}{2}(1)(0)} - \cancelto{0}{\frac{1}{2}(0)(1)} - \frac{1}{2}(1) - \frac{1}{2}(1) +$

$\cancelto{0}{\frac{1}{2}(0)(0)} - \cancelto{0}{\frac{1}{2}(0)\ln(1)} + \cancelto{0}{\frac{1}{2}(1)(0)}$

$$c_1 - c_2 + \frac{1}{2}\int_0^1 e^{-s}\sec s\,ds = 1$$

$\frac{1}{2}\int_0^1 e^{-s}\sec s\,ds \approx .369064975$ or $.3691$

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7 cont'd

$$c_1 + c_2 + .3691 = 2$$

$$c_1 + c_3 + .3691 + 1 = 1$$

$$c_1 - c_2 + .3691 = 1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1.6309 \\ 1 & 0 & 1 & -.3691 \\ 1 & -1 & 0 & .6309 \end{array}\right]$$

$$c_1 \approx 1.1309\,,\quad c_2 = 1/2 \quad c_3 = -3/2$$

$$y(t) = 1.1309\, e^t + 1/2 \cos t - 3/2 \sin t + \tfrac{1}{2} e^t \int_0^t e^{-s} \sec s\, ds + \tfrac{1}{2} t \cos t + \tfrac{1}{2} \cos t \ln|\cos t|$$

$$+ \tfrac{1}{2} t \sin t + \tfrac{1}{2} \sin t \ln|\cos t|$$