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1a. $y'' - y = 0 \quad x_0 = 0$ $\qquad y = \sum_{n=0}^{\infty} a_n x^n, \; y' = \sum_{n=1}^{\infty} a_n n x^{n-1}, \; y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) - a_n\right] x^n = 0$$

Solve for a_{n+2} $\Rightarrow a_{n+2} = \dfrac{a_n}{(n+2)(n+1)}$

$n=0$ $a_2 = \dfrac{a_0}{2\cdot 1} = \dfrac{a_0}{2!}$ $\quad n=1$ $a_3 = \dfrac{a_1}{2\cdot 3} = \dfrac{a_1}{3!}$ $\quad n=2$ $a_4 = \dfrac{a_2}{3\cdot 4} = \dfrac{a_0}{4\cdot 3\cdot 2\cdot 1} = \dfrac{a_0}{4!}$ $\quad n=3$ $a_5 = \dfrac{a_3}{5\cdot 4} = \dfrac{a_1}{5\cdot 4\cdot 3\cdot 2} = \dfrac{a_1}{5!}$

$n=4$ $a_6 = \dfrac{a_4}{6\cdot 5} = \dfrac{a_0}{6!}$ $\quad n=5$ $a_7 = \dfrac{a_5}{7\cdot 6} = \dfrac{a_1}{7!}$

$$y = a_0\left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \dots\right) + a_1\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \dots\right)$$

$$= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = a_0 \cosh x + a_1 \sinh x \leftarrow 3a.$$

b. $y'' + 4y' + 4y = 0 \quad x_0 = 0$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + 4\sum_{n=1}^{\infty} a_n n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} 4a_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) + 4a_{n+1}(n+1) + 4a_n\right] x^n = 0$$

$$a_{n+2} = \frac{-4a_{n+1}(n+1) - 4a_n}{(n+2)(n+1)} = \frac{-4a_{n+1}}{n+2} - \frac{4a_n}{(n+2)(n+1)}$$

$n=0$ $\quad a_2 = \dfrac{-4a_1}{2} - \dfrac{4a_0}{2\cdot 1} = -\frac{1}{2}a_1 - \frac{1}{2}a_0$

$n=1$ $\quad a_3 = \dfrac{-4a_2}{3} - \dfrac{4a_1}{3\cdot 2} = -\frac{4}{3}\left(-\frac{1}{2}a_1 - \frac{1}{2}a_0\right) - \frac{2}{3}a_1 = \cancel{\frac{2}{3}a_1} + \frac{2}{3}a_0 - \cancel{\frac{2}{3}a_1} = \frac{2}{3}a_0$

$n=2$ $\quad a_4 = \dfrac{-4a_3}{4} - \dfrac{4a_2}{4\cdot 3} = -\left(\frac{2}{3}a_0\right) - \frac{1}{3}\left(-\frac{1}{2}a_1 - \frac{1}{2}a_0\right) = -\frac{2}{3}a_0 + \frac{1}{6}a_1 + \frac{1}{6}a_0 = -\frac{1}{2}a_0 + \frac{1}{6}a_1$

$n=3$ $\quad a_5 = -\dfrac{4a_4}{5} - \dfrac{4a_3}{5\cdot 4} = -\frac{4}{5}\left(-\frac{1}{2}a_0 + \frac{1}{6}a_1\right) - \frac{1}{5}\left(\frac{2}{3}a_0\right) = \frac{2}{5}a_0 - \frac{1}{15}a_1 - \frac{2}{15}a_0 = \frac{4}{15}a_0 - \frac{1}{15}a_1$

$$y = a_0\left(1 - \frac{1}{2}x^2 - \frac{2}{3}x^3 - \frac{1}{2}x^4 + \frac{4}{15}x^5 + \dots\right) + a_1\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^4 - \frac{1}{15}x^5 + \dots\right)$$

3c. $y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \quad r = -2$

$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$

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1c. $y'' + xy' + 2y = 0,\ x_0 = 0$

$$\sum_{n=2}^{\infty} a_n(n-1)n\,x^{n-2} + x\sum_{n=1}^{\infty} a_n n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

pull out $n=0$

$$\sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=1}^{\infty} 2a_n x^n + a_2(2)(1)(1) + 2a_0(1) = 0$$

$2a_2 + 2a_0 = 0$

$a_2 = -a_0$

$(n=0)$

$$\sum_{n=1}^{\infty}\Big[a_{n+2}(n+2)(n+1) + \underbrace{a_n n + 2a_n}_{a_n(n+2)}\Big]x^n = 0$$

$$a_{n+2} = \frac{-a_n\cancel{(n+2)}}{\cancel{(n+2)}(n+1)} = \frac{-a_n}{n+1}$$

$n=1$: $a_3 = \dfrac{-a_1}{2}$ $n=2$: $a_4 = \dfrac{-a_2}{3} = \dfrac{-(-a_0)}{3} = \dfrac{a_0}{3}$ $n=3$: $a_5 = \dfrac{-a_3}{4} = \dfrac{-(-\frac{1}{2}a_1)}{4} = \dfrac{1}{8}a_1$

$n=4$: $a_6 = \dfrac{-a_4}{5} = -\dfrac{1}{5}\left(\dfrac{a_0}{3}\right) = -\dfrac{a_0}{15}$ $n=5$: $a_7 = \dfrac{-a_5}{6} = -\dfrac{1}{6}\left(\dfrac{1}{8}a_1\right) = \dfrac{-1}{48}a_1$

$n=6$: $a_8 = \dfrac{-a_6}{7} = -\dfrac{1}{7}\left(\dfrac{-a_0}{15}\right) = \dfrac{1}{105}a_0$ $n=7$: $a_9 = \dfrac{-a_7}{8} = -\dfrac{1}{8}\left(-\dfrac{1}{48}a_1\right) = \dfrac{1}{384}a_1$

$$y = a_0\left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6 + \frac{1}{105}x^8 + \cdots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^7 + \frac{1}{384}x^9 + \cdots\right)$$

alternating signs
odd product in den.
even powers
general term $\dfrac{(-1)^n x^{2n}}{\prod(2n-1)}$
$1\cdot3\cdot5\cdots(2n-1)$

alternating signs
even products in denom
odd powers
$\dfrac{(-1)^n x^{2n+1}}{\prod(2n)}$
$2\cdot4\cdot6\cdot8\cdots(2n)$

neither counts the first term.

1d. $x(x+3)^2y'' - y = 0$

$x(x^2+6x+9)$

$(x^3+6x^2+9x)y'' - y = 0$

$x=0,\ x=-3$ singular points we'll use $x_0=1$

$f(x) = x^3+6x^2+9x$ $f'(x) = 3x^2+12x+9$

$f''(x) = 6x+12$ $f'''(x) = 6$

$f(1) = (1)+6+9 = 16$

$f'(1) = 3(1)+12(1)+9 = 24$

$f''(1) = 6+12 = 18$

$f'''(1) = 6$

$$x^3+6x^2+9x = 16 + 24(x-1) + \frac{18(x-1)^2}{2} + \frac{6(x-1)^3}{6}$$

$$= 16 + 24(x-1) + 9(x-1)^2 + (x-1)^3$$

1d. cont'd

$$\left[16+24(x-1)+9(x-1)^2+(x-1)^3\right]y''-y=0 \qquad y=\sum_{n=0}^{\infty}a_n(x-1)^n$$

$$16\sum_{n=2}^{\infty}a_n n(n-1)(x-1)^{n-2}+24(x-1)\sum_{n=2}^{\infty}a_n n(n-1)(x-1)^{n-2}+9(x-1)^2\sum_{n=2}^{\infty}a_n n(n-1)(x-1)^{n-2}$$

$$+(x-1)^3\sum_{n=2}^{\infty}a_n n(n-1)(x-1)^{n-2}-\sum_{n=0}^{\infty}a_n(x-1)^n=0$$

$$\sum_{n=2}^{\infty}16a_n n(n-1)(x-1)^{n-2}+\sum_{n=2}^{\infty}24a_n n(n-1)(x-1)^{n-1}+\sum_{n=2}^{\infty}9a_n n(n-1)(x-1)^{n}+$$

$$\sum_{n=2}^{\infty}a_n n(n-1)(x-1)^{n+1}-\sum_{n=0}^{\infty}a_n(x-1)^n=0$$

$$\sum_{n=0}^{\infty}16a_{n+2}(n+2)(n+1)(x-1)^n+\sum_{n=1}^{\infty}24a_{n+1}(n+1)n(x-1)^n+\sum_{n=2}^{\infty}9a_n n(n-1)(x-1)^n$$

$$+\sum_{n=3}^{\infty}a_{n-1}(n-1)(n-2)(x-1)^n-\sum_{n=0}^{\infty}a_n(x-1)^n=0$$

$$\sum_{n=3}^{\infty}16a_{n+2}(n+2)(n+1)(x-1)^n+\underset{n=0}{16a_2(2)(1)(1)}+\underset{n=1}{16a_3(3)(2)(x-1)}+\underset{n=2}{16a_4(4)(3)(x-1)^2}$$

$$+\sum_{n=3}^{\infty}24a_{n+1}(n+1)n(x-1)^n+\underset{n=1}{24a_2(2)(1)(x-1)}+\underset{n=2}{24a_3(3)(2)(x-1)^2}+$$

$$\sum_{n=3}^{\infty}9a_n n(n-1)(x-1)^n+\underset{n=2}{9a_2(2)(1)(x-1)^2}+\sum_{n=3}^{\infty}a_{n-1}(n-1)(n-2)(x-1)^n-\sum_{n=3}^{\infty}a_n(x-1)^n$$

$$-\underset{n=0}{a_0(1)}-\underset{n=1}{a_1(x-1)}-\underset{n=2}{a_2(x-1)^2}=0$$

$$\sum_{n=3}^{\infty}\left[16a_{n+2}(n+2)(n+1)+24a_{n+1}(n+1)(n)+9a_n n(n-1)+a_{n-1}(n-1)(n-2)-a_n\right](x-1)^n=0$$

$$32a_2-a_0=0 \quad\Rightarrow\quad a_2=\frac{1}{32}a_0$$

$$96a_3+48a_2-a_1=0 \quad\Rightarrow\quad a_3=\frac{1}{96}a_1-\frac{1}{2}a_2=\frac{1}{96}a_1-\frac{1}{64}a_0$$

$$192a_4+144a_3+\underbrace{18a_2-a_2}_{17a_2}=0 \quad\Rightarrow\quad a_4=-\frac{17a_2}{192}-\frac{3}{4}a_3=\frac{-17}{192}\left(\frac{1}{32}a_0\right)-\frac{3}{4}\left(\frac{1}{96}a_1-\frac{1}{64}a_0\right)$$

$$=\frac{4495}{6144}a_0-\frac{1}{128}a_1$$

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1d cont'd

$$16a_{n+2}(n+2)(n+1)+24a_{n+1}(n+1)n+9a_n n(n-1)+a_{n-1}(n-1)(n-2)-a_n=0$$

$$a_{n+2}=\frac{-24a_{n+1}(n+1)n-9a_n n(n-1)-a_{n-1}(n-1)(n-2)+a_n}{16(n+2)(n+1)}$$

$$a_{n+2}=\frac{-\frac{3}{2}a_{n+1}\cdot n}{(n+2)}-\frac{9}{16}a_n\frac{n(n-1)}{(n+2)(n+1)}-\frac{a_{n-1}}{16}\frac{(n-1)(n-2)}{(n+2)(n+1)}+\frac{a_n}{16(n+2)(n+1)}$$

$n=3$

$$a_5=-\frac{3}{2}\cdot\frac{3}{5}a_4-\frac{9}{16}\cdot\frac{3(2)}{5\cdot 4}a_3-\frac{a_2}{16}\cdot\frac{2\cdot 1}{5\cdot 4}+\frac{a_3}{16(5)(4)}$$

$$=-\frac{9}{10}\left(\frac{4495}{6144}a_0-\frac{1}{128}a_1\right)-\frac{27}{160}\left(\frac{1}{96}a_1-\frac{1}{64}a_0\right)-\frac{1}{160}\left(\frac{1}{32}a_0\right)+\frac{1}{320}\left(\frac{1}{96}a_1-\frac{1}{64}a_0\right)$$

$$a_0\left(-\frac{9}{10}\cdot\frac{4495}{6144}+\frac{27}{160}\cdot\frac{1}{64}-\frac{1}{160}\cdot\frac{1}{32}-\frac{1}{320}\cdot\frac{1}{64}\right)+a_1\left(+\frac{9}{10}\cdot\frac{1}{128}-\frac{27}{160}\cdot\frac{1}{96}+\frac{1}{320}\cdot\frac{1}{96}\right)$$

$$a_0\left(\frac{-3359}{5120}\right)+\frac{163}{30720}a_1$$

$$y=a_0\left(1+\frac{1}{32}x^2-\frac{1}{64}x^3+\frac{4495}{6144}x^4-\frac{3359}{5120}x^5+\dots\right)+$$
$$a_1\left(x-\frac{1}{64}x^3-\frac{1}{128}x^4+\frac{163}{30720}x^5+\dots\right)$$

e. $y''+4y=0 \quad x_0=1 \qquad y=\sum_{n=0}^{\infty}a_n(x-1)^n$

$$\sum_{n=2}^{\infty}a_n(n-1)n(x-1)^{n-2}+4\sum_{n=0}^{\infty}a_n(x-1)^n=0 \Rightarrow \sum_{n=0}^{\infty}a_{n+2}(n+2)(n+1)(x-1)^n+\sum_{n=0}^{\infty}4a_n(x-1)^n=0$$

$$\sum_{n=0}^{\infty}\left[a_{n+2}(n+2)(n+1)+4a_n\right](x-1)^n=0$$

$$a_{n+2}=\frac{-4a_n}{(n+2)(n+1)} \qquad \overset{n=0}{a_2}=\frac{-4a_0}{2\cdot 1}=-2a_0 \qquad \overset{n=1}{a_3}=\frac{-4a_1}{3\cdot 2}=-\frac{2}{3}a_1$$

$n=2$ $$a_4=-\frac{4a_2}{4\cdot 3}=-\frac{1}{3}(-2a_0)=\frac{2}{3}a_0 \qquad \overset{n=3}{a_5}=\frac{-4a_3}{5\cdot 4}=-\frac{1}{5}\left(-\frac{2}{3}a_1\right)=\frac{2}{15}a_1$$

$n=4$ $$a_6=\frac{-4a_4}{6\cdot 5}=-\frac{2}{15}\left(\frac{2}{3}a_0\right)=\frac{-4}{45}a_0 \qquad \overset{n=5}{a_7}=\frac{-4a_5}{7\cdot 6}=-\frac{2}{21}\left(\frac{2}{15}a_1\right)=\frac{-4}{315}a_1$$

1e cont'd

$$y = a_0\left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots\right) + a_1\left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \frac{4}{315}x^7 + \cdots\right)$$

1f. $y'' - y = 0,\ x_0 = 3$ $\qquad y = \sum_{n=0}^{\infty} a_n (x-3)^n$

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-3)^{n-2} - \sum_{n=0}^{\infty} a_n (x-3)^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-3)^n - \sum_{n=0}^{\infty} a_n (x-3)^n = 0$$

$$\sum_{n=0}^{\infty}\left[a_{n+2}(n+2)(n+1) - a_n\right](x-3)^n = 0 \qquad a_{n+2} = \frac{a_n}{(n+2)(n+1)}$$

$n=0$: $a_2 = \frac{a_0}{2\cdot 1}$ $\quad$ $n=1$: $a_3 = \frac{a_1}{3\cdot 2} = \frac{1}{6}a_1$ $\quad$ $n=2$: $a_4 = \frac{a_2}{4\cdot 3} = \frac{1}{12}\left(\frac{1}{2}a_0\right) = \frac{1}{24}a_0$ $\quad$ $n=3$: $a_5 = \frac{a_3}{5\cdot 4} = \frac{1}{20}\left(\frac{1}{6}a_1\right) = \frac{a_1}{120}$

$n=4$: $a_6 = \frac{a_4}{6\cdot 5} = \frac{1}{30}\left(\frac{1}{24}a_0\right) = \frac{1}{720}a_0$ $\quad$ $n=5$: $a_7 = \frac{a_5}{7\cdot 6} = \frac{1}{42}\left(\frac{1}{120}a_1\right) = \frac{a_1}{7!}$ $\quad$ $n=6$: $a_8 = \frac{a_6}{8\cdot 7} = \frac{a_0}{8!}$

$$y = a_0\left(1 + \frac{1}{2!}(x-3)^2 + \frac{1}{4!}(x-3)^4 + \frac{1}{6!}(x-3)^6 + \frac{1}{8!}(x-3)^8 + \cdots\right) + a_1\left((x-3) + \frac{1}{3!}(x-3)^3 + \frac{1}{5!}(x-3)^5 + \frac{1}{7!}(x-3)^7 + \cdots\right)$$

$$= a_0\sum_{n=0}^{\infty}\frac{(x-3)^{2n}}{(2n)!} + a_1\sum_{n=0}^{\infty}\frac{(x-3)^{2n+1}}{(2n+1)!}$$

1g. $xy'' + y' + xy = 0 \quad x_0 = 1 \qquad [(x-1)+1]y'' + y' + [(x-1)+1]y = 0$

$$(x-1)y'' + y'' + y' + (x-1)y + y = 0 \qquad y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$(x-1)\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=1}^{\infty} a_n n(x-1)^{n-1} + (x-1)\sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-1} + \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=1}^{\infty} a_n n(x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=1}^{\infty} a_{n+1}(n+1)n(x-1)^n + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n + \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-1)^n + \sum_{n=1}^{\infty} a_{n-1}(x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=1}^{\infty} a_{n+1}(n+1)n(x-1)^n + \sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n + a_2(2)(1)(1) + \sum_{n=1}^{\infty} a_{n+1}(n+1)(x-1)^n + a_1(1)(1) + \sum_{n=1}^{\infty} a_{n-1}(x-1)^n + \sum_{n=1}^{\infty} a_n (x-1)^n + a_0(1)$$

$$\sum_{n=1}^{\infty}\left[a_{n+1}(n+1)n + a_{n+2}(n+2)(n+1) + a_{n+1}(n+1) + a_{n-1} + a_n\right](x-1)^n + 2a_2 + a_1 + a_0 = 0$$

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1g cont'd

$2a_2 = -a_1 - a_0 \Rightarrow a_2 = -\frac{1}{2}a_1 - \frac{1}{2}a_0$ $n=0$

$a_{n+1}(n+1)n + a_{n+2}(n+2)(n+1) + a_{n+1}(n+1) + a_{n-1} + a_n = 0$

$$a_{n+2} = \frac{-a_n - a_{n-1} - a_{n+1}(n+1) - a_{n+1}(n+1)n}{(n+2)(n+1)}$$

$$= \frac{-a_n}{(n+2)(n+1)} - \frac{a_{n-1}}{(n+2)(n+1)} - \frac{a_{n+1}(1+n)\cancel{(n+1)}}{(n+2)\cancel{(n+1)}}$$

$n=1$

$a_3 = \frac{-a_1}{3\cdot 2} - \frac{a_0}{3\cdot 2} - \frac{a_2(2)}{3} = -\frac{1}{6}a_1 - \frac{1}{6}a_0 - \frac{1}{3}\left(-\frac{1}{2}a_1 - \frac{1}{2}a_0\right) =$

$-\frac{1}{6}a_1 - \frac{1}{6}a_0 + \frac{1}{3}a_1 + \frac{1}{3}a_0 = \frac{1}{6}a_1 + \frac{1}{6}a_0$

$n=2$

$a_4 = \frac{-a_2}{4\cdot 3} - \frac{a_1}{4\cdot 3} - \frac{a_3(3)}{4} = -\frac{1}{12}\left(-\frac{1}{2}a_1 - \frac{1}{2}a_0\right) - \frac{1}{12}a_1 - \frac{3}{4}\left(\frac{1}{6}a_1 + \frac{1}{6}a_0\right) =$

$-\frac{1}{12}a_0 - \frac{1}{6}a_1$

$n=3$

$a_5 = \frac{-a_3}{5\cdot 4} - \frac{a_2}{5\cdot 4} - \frac{a_4\cdot 4}{5} = -\frac{1}{20}\left(\frac{1}{6}a_1 + \frac{1}{6}a_0\right) - \frac{1}{20}\left(-\frac{1}{2}a_1 - \frac{1}{2}a_0\right) - \frac{4}{5}\left(-\frac{1}{12}a_0 - \frac{1}{6}a_1\right)$

$= 1/12\, a_0 - \frac{3}{20}a_1$

$$y = a_0\left(1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{12}(x-1)^5 + \dots\right) +$$

$$a_1\left((x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 - \frac{3}{20}(x-1)^5 + \dots\right)$$

1h. $(x^2+1)y'' + 2xy' = 0 \qquad x_0 = 0$

$$x^2\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + 2x\sum_{n=1}^{\infty} a_n n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} 2a_n n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} 2a_n n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=2}^{\infty} a_{n+2}(n+2)(n+1)x^n + a_2(2)(1)(1) + a_3(3)(2)x + \sum_{n=2}^{\infty} 2a_n n x^n + 2a_1(1)x = 0$$

$$\sum_{n=2}^{\infty}\left[a_n n(n-1) + a_{n+2}(n+2)(n+1) + 2a_n n\right]x^n + 2a_2 + 6a_3x + 2a_1x = 0$$

$a_2 = 0 \qquad 6a_3 + 2a_1 = 0 \Rightarrow a_3 = \frac{-2a_1}{6} = -\frac{1}{3}a_1$

$n=0 \qquad\qquad n=1$

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1h. cont'd

$$a_n n(n-1) + a_{n+2}(n+2)(n+1) + 2a_n n = 0$$

$$a_{n+2} = \frac{-2a_n n - a_n n(n-1)}{(n+2)(n+1)} = \frac{-a_n(2n+n^2-n)}{(n+2)(n+1)} = \frac{-a_n(n^2+n)}{(n+2)(n+1)}$$

$$= \frac{-a_n(\cancel{n+1})n}{(n+2)(\cancel{n+1})} = \frac{-a_n n}{n+2} = a_{n+2}$$

$n=2$

$$a_4 = \frac{-a_2(2)}{4} = \frac{-2}{4}(0) = 0$$ all even powers $> n=0$ are 0

$n=3$ $a_5 = \frac{-a_3(3)}{5} = -\frac{3}{5}\left(-\frac{1}{3}a_1\right) = \frac{1}{5}a_1$ $n=4$ $a_6 = \frac{-a_4(4)}{6} = -\frac{2}{3}(0) = 0$

$n=5$ $a_7 = -\frac{a_5(5)}{7} = -\frac{5}{7}\left(\frac{1}{5}a_1\right) = -\frac{1}{7}a_1$ $n=7$ $a_9 = \frac{-a_7(7)}{9} = \frac{-7}{9}\left(-\frac{1}{7}a_1\right) = \frac{1}{9}a_1$

$$y = a_0(1) + a_1\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \dots\right)$$

$$a_0 + a_1\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(n+1)}$$

2. a. $2xy'' - y' + 2y = 0$

first find the analytic solutions to check your work. 2a. has none. (if you can)

b. $x^3y''' - 6y = 0$ → also 3d.

This is Cauchy-Euler

$y = t^n$ $y' = nt^{n-1}$ $y'' = n(n-1)t^{n-2}$

$y''' = n(n-1)(n-2)t^{n-3}$

auxiliary equation: $[n(n-1)(n-2) - 6]t^n = 0$

$n(n^2-3n+2) - 6 =$

$n^3 - 3n^2 + 2n - 6 = 0$ find real roots 3rd orders must have one

$n^2(n-3) + 2(n-3) = 0$ you can do this by graphing or factoring

$(n^2+2)(n-3) = 0$

$n = \pm\sqrt{2}i, 3$

if by graphing, you'll then need to divide to obtain the quadratic for the quadratic formula

$$y(x) = c_1 t^3 + c_2\cos(\sqrt{2}\ln t) + c_3\sin(\sqrt{2}\ln t)$$

terminating power series

non-terminating power series

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2c. $x^2 y'' + 5xy' + 4y = 0$ Cauchy-Euler $y = t^n$

$n(n-1) + 5n + 4 = 0$

$n^2 - n + 5n + 4 = 0$

$n^2 + 4n + 4 = 0$

$(n+2)^2 = 0$

$n = -2$ repeated

$y = c_1 t^{-2} + c_2 t^{-2} \ln t$ also 3b

both are non-terminating series

2a. assume $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ $\quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$ $\quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$2x \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^{n+r-1} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=-1}^{\infty} 2a_{n+1} (n+r+1)(n+r) x^{n+r} - \sum_{n=-1}^{\infty} a_{n+1} (n+r) x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2a_{n+1} (n+r+1)(n+r) x^{n+r} + 2a_0 (r)(r-1) x^{r-1} - \sum_{n=0}^{\infty} a_{n+1} (n+r) x^{n+r} - a_0 (r-1) x^{r-1}$$
$$+ \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[2a_{n+1} (n+r+1)(n+r) - a_{n+1} (n+r) + 2a_n \right] x^{n+r} + a_0 \left[2r(r-1) - (r-1) \right] x^{r-1} = 0$$

$2r^2 - 2r - r + 1 = 0$

$2r^2 - 3r + 1 = 0$

$(2r-1)(r-1) = 0$

$r = 1/2, \ r = 1$

$r_1 = 1/2$

$$\sum_{n=0}^{\infty} \left[2a_{n+1} (n+3/2)(n+1/2) - a_{n+1} (n+1/2) + 2a_n \right] = 0$$

$a_{n+1} \left[2(n+3/2)(n+1/2) - (n+1/2) \right] = -2a_n$

$a_{n+1} \left\{ (n+1/2) \left[\underbrace{(2n+3) - 1}_{2n+2} \right] \right\} = -2a_n$

$$a_{n+1} = \frac{-2a_n}{(n+1/2)\,2(n+1)} = \frac{-2a_n}{(2n+1)(n+1)}$$

$n=0$: $a_1 = \dfrac{-2a_0}{1} = -2a_0$

$n=1$: $a_2 = \dfrac{-2a_1}{3 \cdot 2} = -\frac{1}{3}(-2a_0) = \frac{2}{3} a_0$

$n=2$: $a_3 = \dfrac{-2a_2}{5 \cdot 3} = -\frac{2}{15}\left(\frac{2}{3}\right) a_0 = \frac{-4}{45} a_0$

$$y_1 = a_0 \left(1 - 2x + \tfrac{2}{3} x^2 - \tfrac{4}{45} x^3 + \dots \right) x^{1/2}$$

$r_2 = 1$

$$\sum_{n=0}^{\infty} \left[2a_{n+1} (n+2)(n+1) - a_{n+1} (n+1) + 2a_n \right] = 0$$

2a cont'd

$$a_{n+1}\left[2(n+2)(n+1)-(n+1)\right]=-2a_n$$

$$a_{n+1}\left[2(n+2)-1\right](n+1)=-2a_n$$

$$a_{n+1}\left[2n+4-1\right](n+1)=-2a_n$$

$$a_{n+1}=\frac{-2a_n}{(2n+3)(n+1)}$$

$n=0$: $a_1=\dfrac{-2a_0}{3\cdot 1}=-\dfrac{2}{3}a_0$

$n=1$: $a_2=\dfrac{-\cancel{2}a_1}{5\cdot\cancel{2}}=\dfrac{-\left(-\frac{2}{3}a_0\right)}{5}=\dfrac{2}{15}a_0$

$n=2$: $a_3=\dfrac{-2a_2}{7\cdot 3}=-\dfrac{2}{21}\left(\dfrac{2}{15}\right)a_0=\dfrac{-4}{315}a_0$

$$y_2=a_0\left(1-\tfrac{2}{3}x+\tfrac{2}{15}x^2-\tfrac{4}{315}x^3+\dots\right)x^1$$

$$y(x)=c_1\left(1-2x+\tfrac{2}{3}x^2-\tfrac{4}{45}x^3+\dots\right)x^{1/2}+c_2\left(1-\tfrac{2}{3}x+\tfrac{2}{15}x^2-\tfrac{4}{315}x^3+\dots\right)x$$

$$=c_1\left(x^{1/2}-2x^{3/2}+\tfrac{2}{3}x^{5/2}-\tfrac{4}{45}x^{7/2}+\dots\right)+c_2\left(x-\tfrac{2}{3}x^2+\tfrac{2}{15}x^3-\tfrac{4}{315}x^4+\dots\right)$$

2b. $x^3y'''-6y=0$ $\quad y=\sum_{n=0}^{\infty}a_nx^{n+r}$ $\quad y'=\sum_{n=0}^{\infty}a_n(n+r)x^{n+r-1}$

$y''=\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r-2}$ $\quad y'''=\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)(n+r-2)\cdot x^{n+r-3}$

$$x^3\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)(n+r-2)x^{n+r-3}-6\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)(n+r-2)x^{n+r}-\sum_{n=0}^{\infty}6a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[a_n(n+r)(n+r-1)(n+r-2)-6a_n\right]x^{n+r}=0$$

no special condition on r here

let $r=0$

$$a_nn(n-1)(n-2)-6a_n=0$$

$$a_n\left[n(n-1)(n-2)-6\right]=0$$

$$n^3-3n^2+2n-6=0$$

$$n^2(n-3)+2(n-6)=0$$

$$(n^2+2)(n-3)=0$$

$$n=3$$

x^3 is a solution

use reduction of order

2b cont'd

$y_2 = v y_1 = v x^3$ $\quad y_2' = v'x^3 + 3vx^2$ $\quad y_2'' = v'' + v' \cdot 6x^2 + v \cdot 6x$

$y_2''' = v''' + v'' \cdot 3x^2 \cdot 3 + v' \cdot 3 \cdot 6x + v \cdot 6$

$$x^3(v''' + v'' \cdot 9x^2 + v' \cdot 6x + 6v) - 6vx^3 = 0$$

$$\frac{v'''x^3 + v'' \cdot 9x^5 + v' \cdot 6x^4}{x^3} + \cancel{6vx^3 - 6vx^3} = 0$$

$$v''' + 9x^2 v'' + 6x v' = 0$$

$u = v'' \Rightarrow$

$u'' + 9x^2 u' + 6xu = 0$

(0 is no longer a singular point)

$u = \sum_{n=0}^{\infty} a_n x^n \quad u' = \sum_{n=1}^{\infty} a_n x^{n-1} \cdot n$

$u'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + 9x^2 \sum_{n=1}^{\infty} a_n n x^{n-1} + 6x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1} 9a_n n x^{n+1} + \sum_{n=0} 6a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=2}^{\infty} 9a_{n-1}(n-1)x^n + \sum_{n=1}^{\infty} 6a_{n-1}x^n = 0$$

$$a_2(2)(1)(1) + a_3(3)(2)x + \sum_{n=2}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=2}^{\infty} 9a_{n-1}(n-1)x^n + 6a_0 x + \sum_{n=2}^{\infty} 6a_{n-1}x^n = 0$$

$$\sum_{n=2}^{\infty}\left[a_{n+2}(n+2)(n+1) + 9a_{n-1}(n-1) + 6a_{n-1}\right]x^n + 2a_2 + (6a_3 + 6a_0)x = 0$$

$a_2 = 0 \quad a_3 = -a_0$

$$a_{n+2}(n+2)(n+1) + a_{n-1}[9(n-1)+6] = 0$$

$9n-9+6$

$9n-3$

$3(3n-1)$

$$a_{n+2} = \frac{-3(3n-1)a_{n-1}}{(n+2)(n+1)}$$

$n=2$

$$a_4 = \frac{-\cancel{3}(5)a_1}{4 \cdot \cancel{3}} = -\frac{5}{4}a_1$$

$n=3$

$$a_5 = \frac{-3(8)a_2}{5 \cdot 4} = 0$$

$n=4$

$$a_6 = \frac{-3(11)a_3}{6 \cdot 5} = \frac{-11}{10}(-a_0) = \frac{11}{10}a_0$$

$n=5$

$$a_7 = \frac{-\cancel{3(14)}a_4}{\cancel{7 \cdot 6}} = -\left(-\frac{5}{4}a_1\right) = \frac{5}{4}a_1$$

2b cont'd.

$n=6$
$$a_8 = \frac{-3(17)a_5}{8\cdot 7} = 0$$

$n=7$
$$a_9 = \frac{-3(20)a_6}{9\cdot 8} = \frac{-5}{6}\left(\frac{11}{10}a_0\right) = \frac{-11}{12}a_0$$

$n=8$
$$a_{10} = \frac{-3(23)a_7}{10\cdot 9} = \frac{-23}{30}\left(\frac{-5}{4}a_1\right) = \frac{23}{24}a_1$$

$$u = a_0\left(1 - 1x^3 + \frac{11}{10}x^6 - \frac{11}{12}x^9 + \cdots\right) + a_1\left(x - \frac{5}{4}x^4 + \frac{5}{4}x^7 + \frac{23}{24}x^{10} + \cdots\right)$$
$= v'$

$$v = a_0\left(x - \frac{1}{5}x^4 + \frac{11}{70}x^7 - \frac{11}{120}x^{10} + \cdots\right) + a_1\left(\frac{1}{2}x^2 - \frac{1}{4}x^5 + \frac{5}{32}x^8 + \frac{23}{264}x^{10} + \cdots\right)$$

$$y_2 = x^3 v$$

$$y(x) = c_1 x^3 + c_2\left(x^4 - \frac{1}{5}x^7 + \frac{11}{70}x^{10} - \frac{11}{120}x^{13} + \cdots\right) + c_3\left(\frac{1}{2}x^5 - \frac{1}{4}x^8 + \frac{5}{32}x^{11} + \frac{23}{264}x^{13} + \cdots\right)$$

2e. $x^2y'' + 5xy' + 4y = 0$ $\qquad y = \sum_{n=0}^{\infty} a_n x^{n+r}$ $\quad y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$

$$y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$$

$$x^2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} + 5x\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + 4\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} 5a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} 4a_n x^{n+r} = 0$$

no condition on r

let $r=0$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} 5a_n(n)x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_n[n(n-1) + 5n + 4]x^n = 0$$

$n^2 - n + 5n + 4 = 0$

$n^2 + 4n + 4 = 0$

$(n+2)^2 = 0$

$n = -2$

$y_1 = x^{-2}$

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2c. Cont'd

$y_2 = v y_1 = v \cdot x^{-2}$ $\quad y_2' = v' x^{-2} - 2v x^{-3}$ $\quad y_2'' = v'' x^{-2} - 4v' x^{-3} + 6v x^{-4}$

$x^2(v'' \cdot x^{-2} - 4v' x^{-3} + 6v x^{-4}) + 5x(v' x^{-2} - 2v x^{-3}) + 4 v x^{-2} = 0$

$v'' - 4v' x^{-1} + \cancel{6v x^{-2}} + 5x^{-1} v' - \cancel{10v x^{-2}} + \cancel{4v x^{-2}} = 0$

$v'' + v' x^{-1} = 0 \qquad u = v' \quad \Rightarrow \quad u' + u x^{-1} = 0$

$u' = -u x^{-1}$

$\int \frac{du}{u} = \int -x^{-1} dx$

$\ln u = -\ln x + c$

$u = 1/x$

$\int v' = \int 1/x \, dx$

$v = \ln x$

$y_2 = \ln x \cdot x^2$

$y(x) = c_1 x^2 + c_2 x^2 \ln x$

we can't obtain a second series solution at $x=0$ since the second function is not defined at zero.

I will also solve these problems by series methods centred at $x=1$ as an alternative to Frobenius

2a. $2xy'' - y' + 2y = 0$ $\qquad x = (x-1) + 1$

$2x = 2(x-1) + 2$

$[2(x-1) + 2] y'' - y' + 2y = 0$

$y = \sum_{n=0}^{\infty} a_n (x-1)^n \qquad y' = \sum_{n=1}^{\infty} a_n (x-1)^{n-1} n$

$y'' = \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2}$

$2(x-1) \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} +$

$2 \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - \sum_{n=1}^{\infty} a_n n (x-1)^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=2}^{\infty} 2a_n n(n-1)(x-1)^{n-1} + \sum_{n=2}^{\infty} 2a_n n(n-1)(x-1)^{n-2} - \sum_{n=1}^{\infty} a_n n(x-1)^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$

$\sum_{n=1}^{\infty} 2a_{n+1}(n+1)n(x-1)^n + \sum_{n=0}^{\infty} 2a_{n+2}(n+2)(n+1)(x-1)^n - \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-1)^n + \sum_{n=0}^{\infty} 2a_n (x-1)^n = 0$

$\sum_{n=1}^{\infty} \left[2a_{n+1}(n+1)n + 2a_{n+2}(n+2)(n+1) - a_{n+1}(n+1) + 2a_n \right](x-1)^n + 2a_2(2)(1) + a_1(1) + 2a_0 = 0$

$2n^2 + n - 1$

$2n^2 + 2n - n - 1$

$4a_2 + a_1 + 2a_0 = 0$

$a_{n+1}\left[2n(n+1) - (n+1)\right] + 2a_{n+2}(n+2)(n+1) + 2a_n = 0$

$a_2 = -\frac{1}{4} a_1 - \frac{1}{2} a_0$

$a_{n+2} = \dfrac{-2a_n - (2n+1)(n+1)a_{n+1}}{2(n+2)(n+1)} = \dfrac{-a_n}{(n+2)(n+1)} - \dfrac{(2n+1)}{2(n+2)} a_{n+1}$

2a.(cont'd)

n=1

$$a_3 = \frac{-a_1}{3\cdot 2} - \frac{a_2(\cancel{3})}{2(\cancel{3})} = -\frac{1}{6}a_1 - \frac{1}{2}\left(-\frac{1}{4}a_1 - \frac{1}{2}a_0\right) = -\frac{1}{24}a_1 + \frac{1}{4}a_0$$

n=2

$$a_4 = \frac{-a_2}{4\cdot 3} - \frac{a_3(5)}{2(4)} = -\frac{1}{12}\left(-\frac{1}{4}a_1 - \frac{1}{2}a_0\right) - \frac{5}{8}\left(-\frac{1}{24}a_1 + \frac{1}{4}a_0\right) = \frac{3}{64}a_1 - \frac{11}{96}a_0$$

n=3

$$a_5 = -\frac{a_3}{5\cdot 4} - \frac{a_4(7)}{2(5)} = -\frac{1}{20}\left(-\frac{1}{24}a_1 + \frac{1}{4}a_0\right) - \frac{7}{10}\left(\frac{3}{64}a_1 - \frac{11}{96}a_0\right) = \frac{-59}{1920}a_1 + \frac{13}{192}a_0$$

$$y(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{4}x^3 - \frac{11}{96}x^4 + \frac{13}{192}x^5 + \dots\right) + a_1\left(x - \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{3}{64}x^4 - \frac{59}{1920}x^5 + \dots\right)$$

2b. $x^3 = 1 + 3(x-1) + \frac{6}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3$

$1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$

$x^3 \to 1$

$3x^2 \to 3$

$6x \to 6$

$6 \to 6$

$$x^3 y''' - 6y = 0 \qquad y = \sum_{n=0}^{\infty} a_n (x-1)^n \qquad y' = \sum_{n=1}^{\infty} a_n n (x-1)^{n-1}$$

$$[1 + 3(x-1) + 3(x-1)^2 + (x-1)^3]y'' - 6y = 0 \qquad y'' = \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2}$$

$$y''' = \sum_{n=3}^{\infty} a_n n(n-1)(n-2)(x-1)^{n-3}$$

$$1\sum_{n=3}^{\infty} a_n n(n-1)(n-2)(x-1)^{n-3} + 3(x-1)\sum_{n=3}^{\infty} a_n n(n-1)(n-2)(x-1)^{n-3}$$

$$+ 3(x-1)^2\sum_{n=0}^{\infty} a_n n(n-1)(n-2)(x-1)^{n-3} + (x-1)^3\sum_{n=3}^{\infty} a_n(n)(n-1)(n-2)(x-1)^{n-3} - 6\sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=3}^{\infty} a_n n(n-1)(n-2)(x-1)^{n-3} + \sum_{n=3}^{\infty} 3a_n n(n-1)(n-2)(x-1)^{n-2} + \sum_{n=3}^{\infty} 3a_n n(n-1)(n-2)(x-1)^{n-1}$$

$$+ \sum_{n=3}^{\infty} a_n n(n-1)(n-2)(x-1)^n - \sum_{n=0}^{\infty} 6a_n(x-1)^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+3}(n+3)(n+2)(n+1)(x-1)^n + \sum_{n=1}^{\infty} 3a_{n+2}(n+2)(n+1)n(x-1)^n + \sum_{n=2}^{\infty} 3a_{n+1}(n+1)n(n-1)(x-1)^n$$

$$+ \sum_{n=3}^{\infty} a_n n(n-1)(n-2)(x-1)^n - \sum_{n=0}^{\infty} 6a_n(x-1)^n = 0$$

$$a_3(3)(2)(1)(1) + a_4(4)(3)(2)(x-1) + a_5(5)(4)(3)(x-1)^2 + \sum_{n=3}^{\infty} a_{n+3}(n+3)(n+2)(n+1)(x-1)^n$$

$$+ 3a_3(3)(2)(1)(x-1) + 3a_4(4)(3)(2)(x-1)^2 + \sum_{n=3}^{\infty} 3a_{n+2}(n+2)(n+1)(n)(x-1)^n + 3a_3(3)(2)(1)(x-1)^2$$

$$+ \sum_{n=3}^{\infty} 3a_{n+1}(n+1)n(n-1)(x-1)^n + \sum_{n=3}^{\infty} a_n n(n-1)(n-2)(x-1)^n - \sum_{n=3}^{\infty} 6a_n(x-1)^n - 6a_0(1) - 6a_1(x-1) - 6a_2(x-1)^2 = 0$$

26 cont'd

$6a_3 - 6a_0 = 0 \Rightarrow a_3 = a_0$

$a_4(24) + 18a_3 - 6a_1 = 0 \Rightarrow a_4 = \dfrac{6a_1 - 18a_3}{24} = \frac{1}{4}a_1 - \frac{3}{4}a_0$

$60a_5 + 72a_4 + 18a_3 - 6a_2 = 0 \Rightarrow a_5 = \dfrac{6a_2 - 18a_3 - 72a_4}{60} = \frac{1}{10}a_2 - \frac{3}{10}a_0 - \frac{6}{5}\left(\frac{1}{4}a_1 - \frac{3}{4}a_0\right)$

$= \frac{1}{10}a_2 + \frac{3}{5}a_0 - \frac{3}{10}a_1$

$$\sum_{n=3}^{\infty}\left[a_{n+3}(n+3)(n+2)(n+1) + 3a_{n+2}(n+2)(n+1)n + 3a_{n+1}(n+1)n(n-1) + a_n n(n-1)(n-2) - 6a_n\right](x-1)^n = 0$$

$n^3 - 3n^2 + 2n - 6$

$(n^2+2)(n-3)\,a_n$

$$a_{n+3} = \frac{-3a_{n+2}(n+2)(n+1)n - 3a_{n+1}(n+1)(n)(n-1) - a_n(n^2+2)(n-3)}{(n+3)(n+2)(n+1)}$$

$$a_{n+3} = -\frac{3n\,a_{n+2}}{n+3} - \frac{3n(n-1)a_{n+1}}{(n+3)(n+2)} - \frac{a_n(n^2+2)(n-3)}{(n+3)(n+2)(n+1)}$$

$n=3$

$$a_6 = -\frac{3\cancel{(3)}a_5}{\cancel{6}_2} - \frac{3(3)\cancel{(2)}a_4}{\cancel{6}\cdot 5} - \cancel{\frac{a_3(11)(0)}{6\cdot5\cdot4}} = -\frac{3}{2}\left(\frac{1}{10}a_2 + \frac{3}{5}a_0 - \frac{3}{10}a_1\right) - \frac{3}{5}\left(\frac{1}{4}a_1 - \frac{3}{4}a_0\right)$$

$$= -\frac{3}{20}a_2 - \frac{9}{20}a_0 + \frac{3}{10}a_1$$

$n=4$

$$a_7 = -\frac{3(4)a_6}{7} - \frac{3\overset{2}{\cancel{(4)}}\cancel{(3)}a_5}{7\cdot\cancel{6}} - \frac{a_4(\overset{3}{\cancel{18}})(1)}{(7)\cancel{6}\cdot5} = -\frac{12}{7}\left(-\frac{3}{20}a_2 - \frac{9}{20}a_0 + \frac{3}{10}a_1\right) - \frac{6}{7}\left(\frac{1}{10}a_2 + \frac{3}{5}a_0 - \frac{3}{10}a_1\right)$$

$$-\frac{3}{35}\left(\frac{1}{4}a_1 - \frac{3}{4}a_0\right) =$$

$$\frac{6}{35}a_2 + \frac{9}{28}a_0 - \frac{39}{140}a_1$$

$$y(x) = a_0\left(1 + (x-1)^3 - \frac{3}{4}(x-1)^4 + \frac{3}{5}(x-1)^5 - \frac{9}{20}(x-1)^6 + \frac{9}{28}(x-1)^7 + \cdots\right) + a_1\left((x-1) + \frac{1}{4}(x-1)^4 + \right.$$

$$\left. -\frac{3}{10}(x-1)^5 + \frac{3}{10}(x-1)^6 - \frac{39}{140}(x-1)^7 + \cdots\right) + a_2\left((x-1)^2 + \frac{1}{10}(x-1)^5 - \frac{3}{20}(x-1)^6\right.$$

$$\left. + \frac{6}{35}(x-1)^7 + \cdots\right)$$

2c. $x^2y'' + 5xy' + 4y = 0$

x^2

$2x/1!$

$2/2!$

$[1 + 2(x-1) + (x-1)^2]y'' + 5[(x-1)+1]y' + 4y = 0$

$y = \sum_{n=0}^{\infty} a_n(x-1)^n$

$y' = \sum_{n=1}^{\infty} a_n n(x-1)^{n-1}$

$y'' = \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2}$

2c cont'd

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + 2(x-1)\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + (x-1)^2\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2}$$

$$+5(x-1)\sum_{n=1}^{\infty} a_n n(x-1)^{n-1} + 5\sum_{n=1}^{\infty} a_n n(x-1)^{n-1} + 4\sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=2}^{\infty} 2a_n n(n-1)(x-1)^{n-1} + \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^n + \sum_{n=1}^{\infty} 5a_n n(x-1)^n$$

$$+\sum_{n=1}^{\infty} 5a_n n(x-1)^{n-1} + \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n + \sum_{n=1}^{\infty} 2a_{n+1}(n+1)n(x-1)^n + \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^n + \sum_{n=1}^{\infty} 5a_n n(x-1)^n$$

$$+\sum_{n=0}^{\infty} 5a_{n+1}(n+1)(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$a_2(2)(1)(1) + a_3(3)(2)(x-1) + \sum_{n=2}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n + \sum_{n=2}^{\infty} 2a_{n+1}(n+1)n(x-1)^n +$$

$$2a_2(2)(1)(x-1) + \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^n + 5a_1(1)(x-1) + \sum_{n=2}^{\infty} 5a_n n(x-1)^n + 5a_1(1)(1)$$

$$+ 5a_2(2)(x-1) + \sum_{n=2}^{\infty} 5a_{n+1}(n+1)(x-1)^n + a_0(1) + a_1(x-1) + \sum_{n=2}^{\infty} a_n(x-1)^n = 0$$

$$2a_2 + 5a_1 + a_0 = 0 \Rightarrow \quad a_2 = -\frac{5a_1}{2} - \frac{a_0}{2}$$

$$6a_3 + 4a_2 + 5a_1 + 10a_2 + a_1 = 0 \Rightarrow 6a_3 + 14a_2 + 6a_1 = 0$$

$$a_3 = -\frac{\cancel{14}^{7}a_2}{\cancel{6}_{3}} - \frac{\cancel{6}a_1}{\cancel{6}} = -\frac{7}{3}\left(-\frac{5}{2}a_1 - \frac{1}{2}a_0\right) - a_1$$

$$= \frac{35}{6}a_1 - a_1 + \frac{7}{6}a_0 = \frac{29}{6}a_1 + \frac{7}{6}a_0$$

$$\sum_{n=2}^{\infty}\left[a_{n+2}(n+2)(n+1) + 2a_{n+1}(n+1)n + a_n n(n-1) + 5a_n n + 5a_{n+1}(n+1) + a_n\right](x-1)^n = 0$$

$$a_{n+2}(n+2)(n+1) + a_{n+1}\underbrace{\left[2n^2+2n+5n+5\right]}_{2n^2+7n+5} + a_n\underbrace{\left[n^2-n+5n+1\right]}_{n^2+4n+1} = 0$$

$$a_{n+2}(n+2)(n+1) + a_{n+1}(2n+5)(n+1) + a_n(n^2+4n+1) = 0$$

$$a_{n+2} = -\frac{a_{n+1}(2n+5)}{(n+2)} - \frac{a_n(n^2+4n+1)}{(n+2)(n+1)}$$

2c cont'd

$n=2$

$$a_4 = \frac{-a_3(9)}{4} - \frac{a_2(13)}{(4)(3)} = \frac{-9}{4}\left(\frac{29}{6}a_1 + \frac{7}{6}a_0\right) - \frac{13}{12}\left(-\frac{5}{2}a_1 - \frac{1}{2}a_0\right) = \frac{-49}{6}a_1 \frac{-25}{12}a_0$$

$n=3$

$$a_5 = -\frac{a_4(11)}{5} - \frac{a_3(\cancel{22}^{11})}{5\cdot\cancel{4}_2} = -\frac{11}{5}\left(\frac{-49}{6}a_1 - \frac{25}{12}a_0\right) - \frac{11}{10}\left(\frac{29}{6}a_1 + \frac{7}{6}a_0\right) = \frac{253}{20}a_1 + \frac{33}{10}a_0$$

$$y(x) = a_0\left(1 - \frac{1}{2}(x-1)^2 + \frac{7}{6}(x-1)^3 - \frac{25}{12}(x-1)^4 + \frac{33}{10}(x-1)^5 + \cdots\right) +$$
$$a_1\left(x - \frac{5}{2}(x-1)^2 + \frac{29}{6}(x-1)^3 - \frac{49}{6}(x-1)^4 + \frac{253}{20}(x-1)^5 + \cdots\right)$$

4a. $y'' + \frac{2}{x}y' + \frac{x+1}{2x^2}y = 0$ $\quad x=0$ singular

$\lim_{x\to 0} \frac{2}{x}\cdot x = 2$ defined $\quad \lim_{x\to 0} \frac{x+1}{2x^2}\cdot x^2 = \lim_{x\to 0}\frac{x+1}{2} = \frac{1}{2}$ defined

$x=0$ is regular

b. $x^2(x-3)^2y'' + (x-3)y' + 5x^2y = 0$

$y'' + \frac{1}{x^2(x-3)}y' + \frac{5}{(x-3)^2}y = 0$ ← Standard form $\quad x=0,\ x=3$ singular

$\lim_{x\to 0} \frac{1}{x^2(x-3)}\cdot x = \lim_{x\to 0}\frac{1}{x(x-3)}$ undefined $\quad x=0$ irregular

$\lim_{x\to 3} \frac{1}{x^2\cancel{(x-3)}}\cdot\cancel{(x-3)} = \lim_{x\to 3}\frac{1}{x^2} = \frac{1}{9}$ defined $\quad \lim_{x\to 3} \frac{5}{\cancel{(x-3)^2}}\cancel{(x-3)^2} = 5$ defined

$x=3$ regular

c. $y'' + \frac{(9-x^2)}{x(1-x)^2}y' + \frac{2+5x}{x^2(1-x)^2}y = 0$ $\quad x=1, x=0$ singular

$\lim_{x\to 0}\frac{9-x^2}{\cancel{x}(1-x)^2}\cdot\cancel{x} = \lim_{x\to 0}\frac{9-x^2}{(1-x)^2} = \frac{9}{1}$ defined $\quad \lim_{x\to 0}\frac{2+5x}{\cancel{x^2}(1-x)^2}\cancel{x^2} = \lim_{x\to 0}\frac{2+5x}{(1-x)^2} = \frac{2}{1}$ defined

$x=0$ is regular

$\lim_{x\to 1}\frac{9-x^2}{x(1-x)^{\cancel{2}}}\cancel{(1-x)} = \lim_{x\to 1}\frac{9-x^2}{x(1-x)}$ not defined

$x=1$ is irregular

4d. $x^3 y'' - (1-\cos x)y' + xy = 0$

$y'' - \frac{(1-\cos x)}{x^3} y' + \frac{1}{x^2} y = 0$ ← standard $x=0$ is singular

$\lim_{x\to 0} \frac{1-\cos x}{x^3} \cdot x = \lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}$ defined

$\lim_{x\to 0} \frac{1}{\cancel{x^2}} \cdot \cancel{x^2} = 1$ defined $x=0$ is regular

e. $3x(x-2)^2 y'' + 2xy' + (x-2)y = 0$

$y'' + \frac{2}{3(x-2)^2} y' + \frac{1}{3x(x-2)} y = 0$ ← standard $x=0, x=2$ singular

$\lim_{x\to 0} \frac{2}{3(x-2)^2} \cdot x =$ defined $\lim_{x\to 0} \frac{1}{3\cancel{x}(x-2)} x^{\cancel{2}} =$ defined $x=0$ regular

$\lim_{x\to 2} \frac{2}{3(x-2)^{\cancel{2}}} \cdot \cancel{(x-2)} = \lim_{x\to 2} \frac{2}{3(x-2)}$ undefined $x=2$ irregular

f. $(x^2+1)x^2 y'' + 2(x^4-1)y' - 12(x-1)^3 y = 0$

$y'' + \frac{2(x^2-1)\cancel{(x^2+1)}}{x^2\cancel{(x^2+1)}} y' - \frac{12(x-1)^3}{x^2(x^2+1)} y = 0$ $x=0$ is singular

$\lim_{x\to 0} \frac{2(x^2-1)}{x^{\cancel{2}}} \cancel{x} =$ undefined $x=$ irregular

g. $y'' + \frac{2x}{1+2x} y' - \frac{3}{1+2x} y = 0$ $x=-1/2$ is singular

$\lim_{x\to -1/2} \frac{2x}{\cancel{1+2x}} \cancel{(1+2x)} =$ defined $\lim_{x\to -1/2} \frac{-3}{\cancel{(1+2x)}} (1+2x)^{\cancel{2}} =$ defined $x=-\frac{1}{2}$ is regular

h. $(x\sin x)y'' + (\cos x)y' + (\ln x)y = 0$

$y'' + \frac{\cos x}{x\sin x} y' + \frac{\ln x}{x\sin x} y = 0$ standard $x=0$ is singular

$\lim_{x\to 0} \frac{\cos x}{\cancel{x}\sin x} \cdot \cancel{x} = \lim_{x\to 0} \cot x$ undefined $x=0$ is irregular

4i. $y'' + \frac{1}{x}y' + \frac{1}{x^3}y = 0$ $\quad x=0$ is singular

$\lim_{x\to 0} \frac{1}{\cancel{x}}\cdot\cancel{x} =$ defined $\quad \lim_{x\to 0} \frac{1}{x^3}\cdot x^2 = \lim_{x\to 0}\frac{1}{x}$ undefined

$x=0$ is irregular

j. $x^2(x-3)^2y'' + (x-3)y' + 5x^2y = 0$

$y'' + \frac{1}{x^2(x-3)}y' + \frac{5}{(x-3)^2}y = 0$ ← standard $\quad x=0, x=3$ is singular

$\lim_{x\to 0}\frac{1}{x^{\cancel{2}}(x-3)}\cdot\cancel{x} = \lim_{x\to 0}\frac{1}{x(x-3)}$ not defined $\quad x=0$ is irregular

$\lim_{x\to 3}\frac{1}{x^2\cancel{(x-3)}}\cancel{(x-3)} = \lim_{x\to 3}\frac{1}{x^2} = \frac{1}{9}$ defined $\quad \lim_{x\to 3}\frac{5}{\cancel{(x-3)^2}}\cdot\cancel{(x-3)^2} = 5$ defined

$x=3$ is regular

k. $(1-\ln x)y'' + 2xy' + (x\ln^2 x)y = 0$

$y'' + \frac{2x}{1-\ln x}y' + \frac{x\ln^2 x}{1-\ln x}y = 0$ Standard $\quad x=e$ is singular

$\lim_{x\to e}\frac{2x}{1-\ln x}\cdot(x-e) = \lim_{x\to e}\frac{2x^2-2ex}{1-\ln x} = \lim_{x\to e}\frac{4x-2e}{-1/x}\cdot\frac{x}{x} = \lim_{x\to e}\frac{4x^2-2ex}{-1}$ defined

$\left(\frac{0}{0}\right)$ $\quad$ L'Hopital's

$\lim_{x\to e}\frac{x\ln^2 x}{1-\ln x}(x-e)^2 = \lim_{x\to e}\frac{x^2\ln^2 x - ex\ln^2 x}{1-\ln x} = \lim_{x\to e}\frac{2x\ln^2 x + x^2\cdot 2\ln x\cdot\frac{1}{x} - e\ln^2 x - ex\cdot 2\ln x\cdot\frac{1}{x}}{-\frac{1}{x}}$

$\frac{0}{0}$ $\quad$ l'Hopital's

$\lim_{x\to e} -\left[2x^2\ln^2 x + 2x^2\ln x - ex\ln^2 x - 2ex\ln x\right] =$ defined

$x=e$ is regular

5a. $\vec{u} = \begin{bmatrix}12\\3\\-5\end{bmatrix}, \vec{v}\begin{bmatrix}2\\-3\\3\end{bmatrix}$ $\quad \vec{u}\cdot\vec{v} = 24-9-15=0$ yes, they are orthogonal

b. $\vec{u} = \begin{bmatrix}3\\2\\-5\\0\end{bmatrix}, \vec{v} = \begin{bmatrix}-4\\1\\-2\\6\end{bmatrix}$ $\quad \vec{u}\cdot\vec{v} = -12+2+10+0=0$ yes, they are orthogonal

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5c. $\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$

$\vec{u}_1 \cdot \vec{u}_2 = -2+2-1+1=0 \quad \vec{u}_1 \cdot \vec{u}_3 = 1+2-2-1=0 \quad \vec{u}_1 \cdot \vec{u}_4 = -1+2+1-2=0$

$\vec{u}_2 \cdot \vec{u}_3 = -2+1+2-1=0 \quad \vec{u}_2 \cdot \vec{u}_4 = 2+1-1-2=0 \quad \vec{u}_3 \cdot \vec{u}_4 = -1+1-2+2=0$

yes, they are mutually orthogonal.