MAT 212, Quiz #16, Summer 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Find a series solution to the equation $xy'' + 6y = 0$. You may use either the method of Froebenius ($\cdot\, x^r$) or shift the solution to center around an ordinary point.

Shift $\quad X=(x-1)+1 \qquad y=\sum_{n=0}^{\infty} a_n(x-1)^n$

Froebenius on next page →

$$(x-1)y''+y''+6y=0$$

$$(x-1)\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2}+\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2}+6\sum_{n=0} a_n(x-1)^n=0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-1}+\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2}+\sum_{n=0}^{\infty} 6a_n(x-1)^n=0$$

$$\sum_{n=1}^{\infty} a_{n+1} n(n+1)(x-1)^n+\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n+\sum_{n=0}^{\infty} 6a_n(x-1)^n=0$$

$$\sum_{n=1}^{\infty}\left[a_{n+1}n(n+1)(x-1)^n+a_{n+2}(n+2)(n+1)+6a_n\right](x-1)^n+a_2(2)(1)(1)+6a_0(1)=0$$

$2a_2=-6a_0$

$\Rightarrow a_2=-3a_0$

$$a_{n+2}=\frac{-6a_n-a_{n+1}(n+1)n}{(n+2)(n+1)}=\frac{-6a_n}{(n+2)(n+1)}-\frac{a_{n+1}n}{n+2}$$

$n=1$

$$a_3=-\frac{\cancel{6}a_1}{\cancel{3\cdot 2}}-\frac{a_2\cdot 1}{3}=-a_1-\frac{1}{3}(-3a_0)=-a_1+a_0$$

$n=2$

$$a_4=\frac{-\cancel{6}a_2}{4\cdot\cancel{3}}-\frac{a_3\cdot\cancel{2}}{\cancel{4}\,2}=-\frac{1}{2}(-3a_0)-\frac{1}{2}(-a_1+a_0)=\frac{3}{2}a_0+\frac{1}{2}a_1-\frac{1}{2}a_0=a_0+\frac{1}{2}a_1$$

$n=3$

$$a_5=-\frac{\cancel{6}a_3}{5\cdot\cancel{4}}-\frac{a_4\cdot 3}{5}=-\frac{3}{5}(-a_1+a_0)-\frac{3}{5}\left(a_0+\frac{1}{2}a_1\right)=\frac{3}{5}a_1-\frac{3}{5}a_0-\frac{3}{5}a_0-\frac{3}{10}a_1=\frac{3}{10}a_1-\frac{6}{5}a_0$$

$$y(x)=a_0\left(1-3(x-1)^2+(x-1)^3+(x-1)^4-\frac{6}{5}(x-1)^5+\dots\right)+a_1\left((x-1)-(x-1)^3+\frac{1}{2}(x-1)^4+\frac{3}{10}(x-1)^5+\dots\right)$$

2. Find the indicated dot products, given: $\vec{u}=\begin{bmatrix}1\\4\end{bmatrix}, \vec{v}=\begin{bmatrix}2\\3\end{bmatrix}, \vec{x}=\begin{bmatrix}i\\2-i\end{bmatrix}, \vec{y}=\begin{bmatrix}1+i\\3i\end{bmatrix}$

a. $\vec{u}\cdot\vec{v}$

$2+12=14$

$\begin{bmatrix}1 & 4\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}$

b. $\vec{x}\cdot\vec{y}$

$\begin{bmatrix}-i & 2+i\end{bmatrix}\begin{bmatrix}1+i\\3i\end{bmatrix}=-i+1+6i-3$

$-2+5i$

Frobenius method

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$x \cdot \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + 6 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} 6 a_n x^{n+r} = 0$$

$$\sum_{n=-1}^{\infty} a_{n+1} (n+r+1)(n+r) x^{n+r} + \sum_{n=0}^{\infty} 6 a_n x^{n+r} = 0$$

$$a_0 (r)(r-1) x^{r-1} + \sum_{n=0}^{\infty} \left[a_{n+1} (n+r+1)(n+r) + 6 a_n \right] x^{n+r} = 0$$

$r=0,\ r=1$

$r=0$ $\quad \sum_{n=0}^{\infty} \left[a_{n+1} (n+1)(n) + 6 a_n \right] x^n = 0 \qquad \frac{-6a_n}{(n+1)n} = a_{n+1}$

$r=1$ $\quad \sum_{n=0}^{\infty} \left[a_{n+1} (n+2)(n+1) + 6 a_n \right] x^n = 0 \qquad \frac{-6a_n}{(n+2)(n+1)} = a_{n+1}$

$r=0$ $\quad n=1 \quad \frac{-6a_1}{2(1)} = -3a_1 = a_2$ $\qquad n=0 \quad \frac{-\cancel{6}^{3} a_0}{\cancel{2}\cdot 1} = -3a_0 = a_1$

$n=2 \quad -\frac{\cancel{6}a_2}{\cancel{3\cdot 2}} = -a_2 = -(-3a_1) = 3a_1 = a_3$ $\qquad n=1 \quad -\frac{\cancel{6}a_1}{\cancel{3\cdot 2}} = -a_1 = -(-3a_0) = 3a_0 = a_2$

$n=3 \quad -\frac{\cancel{6}a_3}{\cancel{4\cdot 3}_{2}} = -\frac{1}{2}(3a_1) = -\frac{3}{2}a_1$ $\qquad n=2 \quad -\frac{\cancel{6}a_2}{\cancel{4\cdot 3}_{2}} = -\frac{1}{2}a_2 = -\frac{1}{2}(3a_0) = -\frac{3}{2}a_0$

$$y(x) = c_1\left(x - 3x^2 + 3x^3 - \frac{3}{2}x^4 + \dots\right) + c_2\left(1 - 3x + 3x^2 - \frac{3}{2}x^3 + \dots\right)$$