MATH 240, Exam #2 Practice, Summer 2017 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Find an equation for the plane through the points $A(0,1,1), B(2,2,0), C(3,0,3)$. (12 points)

$\vec{v} = \langle 2,1,-1 \rangle$

$\vec{w} = \langle 1,-2,3 \rangle$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -2 & 3 \end{vmatrix} =$$

$$(3-2)\hat{i} - (6+1)\hat{j} + (-4-1)\hat{k}$$

$$\langle 1,-7,-5 \rangle$$

$$1(x-0) - 7(y-1) - 5(z-1) = 0$$

2. Let $z = x^2 + 3y^2 - 11y$, where $x = 4\sin t, y = 2\cos t$. Find $\frac{dz}{dt}$ and evaluate it at $t = \frac{\pi}{6}$. (12 points)

$\frac{\partial z}{\partial x} = z_x = 2x$

$\frac{\partial z}{\partial y} = z_y = 6y - 11$

$\frac{dx}{dt} = 4\cos t$

$\frac{dy}{dt} = -2\sin t$

$\cos(\pi/6) = \frac{\sqrt{3}}{2}$

$\sin(\pi/6) = \frac{1}{2}$

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

$$\frac{dz}{dt} = 2x(4\cos t) + (6y-11)(-2\sin t)$$

$$= 2(4\sin t)(4\cos t) + [6(2\cos t) - 11](-2\sin t)$$

$$= 8\left(\frac{1}{\cancel{2}}\right)(4)\left(\frac{\sqrt{3}}{\cancel{2}}\right) + \left[\underset{6}{\cancel{12}}\frac{\sqrt{3}}{\cancel{2}} - 11\right]\left(-\cancel{2}\left(\frac{1}{\cancel{2}}\right)\right)$$

$$= 8\sqrt{3} + (6\sqrt{3} - 11)(-1)$$

$$= 2\sqrt{3} + 11$$

3. Given the implicit function $x^5 + 3x^2y^3 + 2y^4 = 0$, find $\frac{dy}{dx}$. (12 points)

$$F = x^5 + 3x^2y^3 + 2y^4$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{5x^4 + 6xy^3}{9x^2y^2 + 8y^3}$$

4. Find the following limits, or prove that they do not exist. (10 points each)

a. $\lim\limits_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1} \cdot \frac{\sqrt{x-y}+1}{\sqrt{x-y}+1} = \lim\limits_{(x,y)\to(2,1)} \frac{\cancel{(x-y-1)}(\sqrt{x-y}+1)}{\cancel{(x-y-1)}} =$

$$\lim_{(x,y)\to(2,1)} \sqrt{x-y}+1 = \sqrt{2-1}+1 = 1+1 = \boxed{2}$$

b. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^2}$ $\qquad y = kx^{3/2}$

$$\lim_{x\to 0} \frac{x^2 \cdot kx^{3/2}}{x^3 + k^2x^3} = \lim_{x\to 0} \frac{\cancel{x^3} \cdot x^{1/2}k}{\cancel{x^3}(1+k^2)} = \lim_{x\to 0} \frac{x^{1/2}k}{1+k^2} = \boxed{0}$$

5. Consider the function $f(x,y) = x^4 + 3xy^2 - y^2 + 12y - 11$. Find the unit vectors that give the direction of the steepest ascent and steepest descent at the point $P(-2,1)$. (12 points)

$$\nabla f = \langle 4x^3 + 3y^2, 6xy - 2y + 12 \rangle$$

$$\langle 4(-2)^3 + 3(1)^2, 6(-2)(1) - 2(1) + 12 \rangle$$

$$\langle -32 + 3, -12 - 2 + 12 \rangle = \langle -29, -2 \rangle$$

$$\|\langle -29, -2 \rangle\| = \sqrt{841 + 4} = \sqrt{845}$$

$$13\sqrt{5}$$

greatest ascent $\left\langle \frac{-29}{13\sqrt{5}}, \frac{-2}{13\sqrt{5}} \right\rangle$

greatest descent $\left\langle \frac{-29}{13\sqrt{5}}, \frac{2}{13\sqrt{5}} \right\rangle$

6. Find an equation of the tangent plane to the surface $x^2 + y^2 + 10z^2 = 540$ at the point $(12,6,-6)$. (15 points)

$$x^2 + y^2 + 10z^2 - 540 = 0 = F$$

$$\nabla F = \langle 2x, 2y + 20z \rangle$$

$$= \langle 2(12), 2(6), 20(-6) \rangle$$

$$= \langle 24, 12, -120 \rangle$$

or $\langle 2, 1, -10 \rangle$

$$2(x-12) + (y-6) - 10(z+6) = 0$$

7. Examine the function $f(x,y) = 4 + x^3 + y^3 - 3xy$ for local max, local min and saddle points. (20 points)

$f_x = 3x^2 - 3y = 0$ $\quad \cancel{3}x^2 = \cancel{3}y \Rightarrow x^2 = y$

$f_y = 3y^2 - 3x = 0$ $\quad \cancel{3}y^2 = \cancel{3}x \Rightarrow y^2 = x$

$(y^2)^2 = y$

$y^4 = y \Rightarrow y^4 - y = 0$

$y(y^3 - 1) = 0$

$y = 0, \; y = 1$

$x = 0 \quad x = 1$

$f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = -3$

$D(0,0) = (0)(0) - (-3)^2 = -9 \overset{<0}{} $ Saddle point

$D(1,1) = (6)(6) - (-3)^2 = 36 - 9 = 27 > 0$

max or min

$f_{xx} > 0 \cup$

minimum

8. Use Lagrange multipliers to optimize the function $f(x,y) = x^2 + y^2 - 2x - 2y + 14$ subject to the constraint $x^2 + y^2 = 16$. (20 points)

$x^2 + y^2 - 16 = 0$

$\nabla f = \lambda \nabla g$

$2x - 2 = \lambda 2x$

$2y - 2 = \lambda 2y$

$(2\sqrt{2}, 2\sqrt{2})$

$(-2\sqrt{2}, -2\sqrt{2})$

$x - 1 = \lambda x \Rightarrow \lambda = \frac{x-1}{x}$

$y - 1 = \lambda y \Rightarrow \lambda = \frac{y-1}{y}$

$\frac{x-1}{x} = \frac{y-1}{y}$

$\cancel{xy} - y = \cancel{xy} - x$

$x = y$

$x^2 + y^2 = 16$

$x^2 + x^2 = 16$

$2x^2 = 16$

$x^2 = 8$

$x = \pm 2\sqrt{2}$