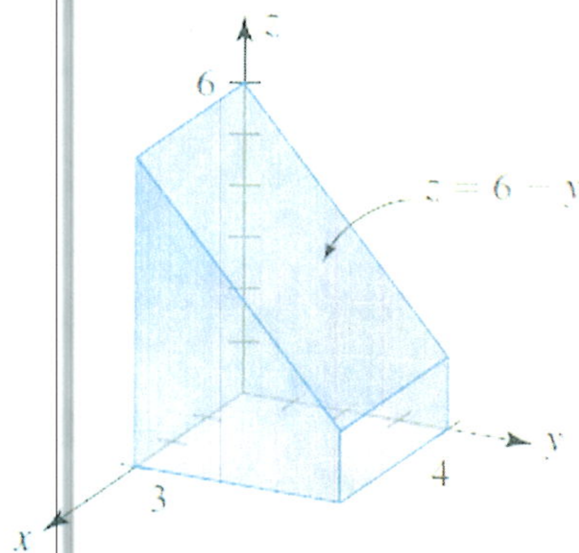
MATH 240, Exam #3, Summer 2017 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Find the volume of the solid under the function $f(x, y) = 6 - y$ on the region bounded by $0 \leq x \leq 3, 0 \leq y \leq 4$ using a double integral. (9 points)



$$\int_0^3 \int_0^4 6 - y \, dy \, dx =$$

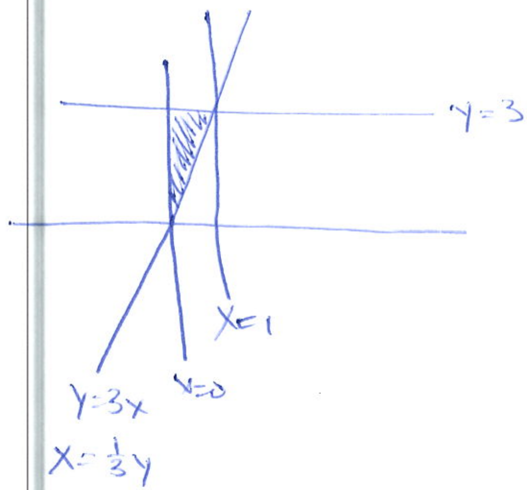
$$\int_0^3 6y - \frac{1}{2}y^2 \Big|_0^4 dx =$$

$$\int_0^3 24 - 8 \, dx = \int_0^3 16 \, dx =$$

$$16x \Big|_0^3 = \boxed{48}$$

2. The integral $\int_0^1 \int_{3x}^{3} 6e^{y^2}\,dydx$ can be evaluated only by changing the order of integration. Sketch the region of integration, reverse the order of integration, and evaluate the integral. (8 points)

$x=1$, $y=3$, $x=0$, $y=3x$



$$\int_0^3 \int_0^{\frac{1}{3}y} 6e^{y^2}\,dx\,dy =$$

$$\int_0^3 6e^{y^2}\cdot x\Big|_0^{\frac{1}{3}y} dy =$$

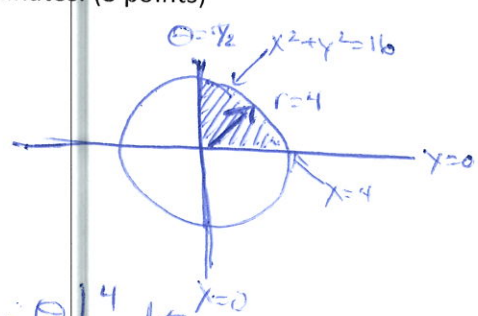
$$\int_0^3 2ye^{y^2}\,dy = e^{y^2}\Big|_0^3 =$$

$$\boxed{e^9 - 1}$$

3. Evaluate $\int_0^4 \int_0^{\sqrt{16-x^2}} ydydx$ by converting to polar coordinates. (8 points)

$y = \sqrt{16-x^2} \rightarrow x^2+y^2=16$, $y=0$

$y = r\sin\theta$



$$\int_0^{\pi/2}\int_0^4 r\sin\theta\, r\,dr\,d\theta =$$

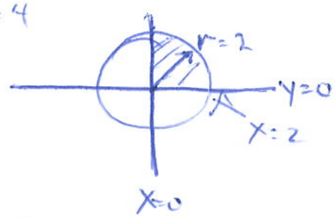
$$\int_0^{\pi/2}\int_0^4 r^2\sin\theta\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{3}r^3\sin\theta\Big|_0^4 d\theta =$$

$$\int_0^{\pi/2} \frac{64}{3}\sin\theta\,d\theta = -\frac{64}{3}\cos\theta\Big|_0^{\pi/2} = -\frac{64}{3}\underbrace{\cos\tfrac{\pi}{2}}_{(0)} + \frac{64}{3}\underbrace{\cos(0)}_{(1)} = \frac{64}{3}$$

4. Evaluate the integral $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{x^2+y^2}} \frac{1}{x^2+y^2}\, dzdydx$ in cylindrical coordinates. (10 points)

$x=2$, $y=\sqrt{4-x^2}$, $z=\sqrt{x^2+y^2} \Rightarrow z=r$

$x=0$, $y=0$, $x^2+y^2=r$



$$\int_0^{\pi/2}\int_0^2\int_0^r \frac{1}{r^{\cancel{2}}}\cdot \cancel{r}\, dz\,dr\,d\theta =$$

$$\int_0^{\pi/2}\int_0^2\int_0^r \frac{1}{r}\, dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 z\cdot\frac{1}{r}\Big|_0^r dr\,d\theta =$$

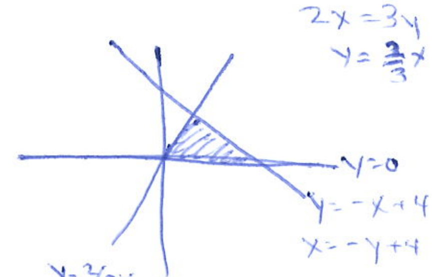
$$\int_0^{\pi/2}\int_0^2 \frac{r}{r}\, dr\,d\theta = \int_0^{\pi/2}\int_0^2 dr\,d\theta = \int_0^{\pi/2} r\Big|_0^2 d\theta = \int_0^{\pi/2} 2\, d\theta$$

$$= 2\theta\Big|_0^{\pi/2} = \boxed{\pi}$$

5. Set up an iterated integral to find the area of the region bounded by $2x-3y=0, x+y=4$, $y=0$. Evaluate the integral to find the area. (10 points)

$2x=3y$

$y=\frac{2}{3}x$



$x=\frac{3}{2}y$

$$\int_0^{8/5}\int_{\frac{3}{2}y}^{-y+4} 1\, dx\, dy = \int_0^{8/5} x\Big|_{3/2y}^{-y+4} dy =$$

$$\int_0^{8/5} -y+4-\frac{3}{2}y\, dy$$

$$\int_0^{8/5} -\frac{5}{2}y+4\, dy = -\frac{5}{4}y^2+4y\Big|_0^{8/5}$$

$\frac{3}{2}y=-y+4$

$3y=-2y+8$

$5y=8$

$y=\frac{8}{5}$

$$-\frac{5}{4}\left(\frac{8}{5}\right)^2+4(8/5)-0=$$

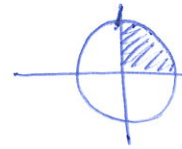
$$-\frac{\cancel{5}}{\cancel{4}}\left(\frac{\cancel{64}^{16}}{\cancel{25}_{5}}\right)+\frac{32}{5}$$

$$-\frac{16}{5}+\frac{32}{5}=\boxed{\frac{16}{5}}$$

6. Use a double integral to find the volume of the indicated solid: $z = x + y, x^2 + y^2 = 1$, first octant. (10 points)

$\hookrightarrow z = x+y \Rightarrow z = r\cos\theta + r\sin\theta$

$$\int_0^{\pi/2}\int_0^1 r(\cos\theta + \sin\theta)\, r\,dr\,d\theta$$



polar/ cylindrical

$$\int_0^{\pi/2} \frac{1}{3}r^3(\cos\theta + \sin\theta)\Big|_0^1 d\theta =$$

$z = r(\cos\theta + \sin\theta)$

$$\frac{1}{3}\int_0^{\pi/2} \cos\theta + \sin\theta\, d\theta = \frac{1}{3}\Big[\sin\theta - \cos\theta\Big|_0^{\pi/2} =$$

$$\frac{1}{3}[1-0-(0-1)] = \boxed{\frac{2}{3}}$$

7. Find the area of the surface given by $f(x, y) = 4 + 2x + 2y$ over the region R: rectangle with vertices $\underbrace{(0,0), (2,0)}_{0 \le x \le 2}, \underbrace{(2,4), (0,4)}_{0 \le y \le 4}$. (10 points)

$f_x = 2,\ f_y = 2$

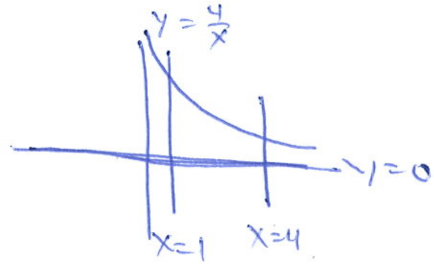
$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

$$\int_0^2\int_0^4 3\,dy\,dx =$$

$$\int_0^2 3y\Big|_0^4 dx = \int_0^2 12\,dx = 12x\Big|_0^2 = \boxed{24}$$

8. Find the mass and centroid of the center of mass of the lamina bounded by the graphs of the equations $xy = 4, x = 1, x = 4, y = 0$ with the density $\rho = ky^2$. (15 points)

$y = \frac{4}{x}$



$$M = \int_1^4 \int_0^{4/x} ky^2\,dy\,dx = \int_1^4 \frac{k}{3} \underbrace{y^3}_{\frac{64}{x^3}} \Big|_0^{4/x} dx =$$

$$\int_1^4 \frac{k\cdot 64}{3x^3}\,dx = \int_1^4 \frac{k64}{3}\cdot x^{-3}\,dx = -\frac{32k}{3} x^{-2}\Big|_1^4 = -\frac{\cancel{32}^{\,2}k}{3}\cdot\frac{1}{\cancel{16}} + \frac{32k}{3}\cdot\frac{1}{1} =$$

$10k$

$$M_x = \int_1^4 \int_0^{4/x} ky^3\,dy\,dx = \int_1^4 \frac{k}{4} y^4 \Big|_0^{4/x} dx = \int_1^4 \frac{k}{\cancel{4}}\cdot\frac{\overset{64}{\cancel{256}}}{x^4}\,dx =$$

$$\int_1^4 64k\,x^{-4}\,dx = -\frac{64k}{3}\cdot x^{-3}\Big|_1^4 = -\frac{\cancel{64}k}{3}\cdot\frac{1}{\cancel{64}} + \frac{64k}{3}\cdot\frac{1}{1} = 21k$$

$$M_y = \int_1^4 \int_0^{4/x} kxy^2\,dy\,dx = \int_1^4 \frac{k}{3} x y^3 \Big|_0^{4/x} = \int_1^4 \frac{k\cdot 64}{3x^3}\cdot x\,dx =$$

$$\int_1^4 \frac{64k}{3}\cdot x^{-2}\,dx = -\frac{64k}{3}\cdot\frac{1}{x}\Big|_1^4 = -\frac{\overset{16}{\cancel{64}}k}{3}\cdot\frac{1}{\cancel{4}} + \frac{64k}{3}\cdot\frac{1}{1} = 16k$$

$$\bar{x} = \frac{M_y}{M} = \frac{16\cancel{k}}{10\cancel{k}} = \frac{16}{10} = \frac{8}{5}$$

$$\bar{y} = \frac{M_x}{M} = \frac{21\cancel{k}}{10\cancel{k}} = \frac{21}{10}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, \frac{21}{10}\right)$$

9. Find integrals for I_x, I_y, I_z for the cube $Q:\{(x,y,z): 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ with density $\rho(x,y,z) = kx^2y^2z^2$. You do not need to integrate them. (10 points)

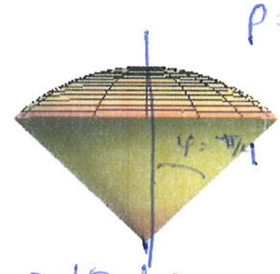
$$I_x = \int_0^1\int_0^1\int_0^1 kx^2y^2z^2(y^2+z^2)\,dz\,dy\,dx$$

$$I_y = \int_0^1\int_0^1\int_0^1 kx^2y^2z^2(x^2+z^2)\,dz\,dy\,dx$$

$$I_z = \int_0^1\int_0^1\int_0^1 kx^2y^2z^2(x^2+y^2)\,dz\,dy\,dx$$

10. Find the volume of the solid between the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 12$. (10 points)

$\rho\cos\varphi = \rho\sin\varphi$ $\Rightarrow \varphi = \pi/4$ $\qquad \rho = \sqrt{12}$



$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sqrt{12}} \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

$$\int_0^{\pi/4}\int_0^{2\pi} \frac{\rho^3}{3}\Big|_0^{\sqrt{12}} \sin\varphi\, d\theta\, d\varphi = \int_0^{\pi/4}\int_0^{2\pi} \frac{\cancel{12}^{\,4}\sqrt{12}}{\cancel{3}} \sin\varphi\, d\theta\, d\varphi =$$

$$\int_0^{\pi/4} 4\sqrt{12}\sin\varphi\, \theta\Big|_0^{2\pi} d\varphi = \int_0^{\pi/4} 8\pi\sqrt{12}\sin\varphi\, d\varphi = -8\pi\sqrt{12}\cos\varphi\Big|_0^{\pi/4}$$

$$= -8\pi\underset{2\sqrt{3}}{\sqrt{12}}\cdot\left(\frac{1}{\sqrt{2}}-1\right) = \boxed{\frac{16\pi\sqrt{3}(\sqrt{2}-1)}{\sqrt{2}}}$$

$x^2+y^2+z^2=9$ $\quad \rho = 3$

11. Convert the integral $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2}\, dz\, dy\, dx$ to spherical coordinates. Evaluate the integral. (10 points)

first octant $\quad \rho \sin\varphi$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho \sin\varphi \cdot \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi =$$

$$\int_0^{\pi/2}\int_0^{\pi/2} \rho^3 \sin^2\varphi\, d\rho\, d\theta\, d\varphi = \int_0^{\pi/2}\int_0^{\pi/2} \frac{\rho^4}{4}\Big|_0^3 \sin^2\varphi\, d\theta\, d\varphi =$$

$$\frac{81}{4}\int_0^{\pi/2}\int_0^{\pi/2} \sin^2\varphi\, d\theta\, d\varphi = \frac{81}{4}\int_0^{\pi/2} \theta \cdot \sin^2\varphi \Big|_0^{\pi/2} d\varphi =$$

$$\frac{81\pi}{8}\int_0^{\pi/2}(\sin^2\varphi)\,d\varphi = \frac{81\pi}{8}\int_0^{\pi/2} \frac{1}{2}(1-\cos 2\varphi)\,d\varphi = \frac{81\pi}{16}\int_0^{\pi/2} 1-\cos 2\varphi\, d\varphi$$

$$= \frac{81\pi}{16}\left[\varphi - \frac{1}{2}\sin 2\varphi\right|_0^{\pi/2} = \frac{81\pi}{16}\left[\pi/2 - \cancel{\tfrac{1}{2}(0)} - 0\right] = \boxed{\frac{81\pi^2}{32}}$$

12. Use the change of variables $x = \frac{1}{2}(u+v), y = \frac{1}{2}(u-v)$ to evaluate the double integral $\iint_R \ln(x+y)\, dA$ over the region bounded by the parallelogram with vertices at $(3,2), (4,3), (3,4), (2,3)$. (13 points)

$$x+y = \tfrac{1}{2}(u+v) + \tfrac{1}{2}(u-v)$$
$$= \tfrac{1}{2}u + \cancel{\tfrac{1}{2}v} + \tfrac{1}{2}u - \cancel{\tfrac{1}{2}v} = u$$

$$\int_5^7\int_{-1}^1 \ln u \cdot \frac{1}{2}\, dv\, du =$$

$$\int_5^7 \frac{1}{2}\ln u \cdot v\Big|_{-1}^1 du = \int_5^7 \ln u\, du =$$

$$u\ln u - \int du = u\ln u - u\Big|_5^7 =$$

$p = \ln u \quad dq = du$
$dp = \frac{1}{u}du \quad q = u$

$$7\ln 7 - 7 - (5\ln 5 - 5) =$$
$$\ln 7^7 - 7 - \ln 5^5 + 5 =$$
$$\boxed{\ln\left(\frac{7^7}{5^5}\right) - 2}$$

$$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}$$

$$= -\frac{1}{4} - \frac{1}{4} \Rightarrow |J| = 1/2$$
$$= -\frac{1}{2}$$

$3 = \frac{1}{2}(u+v) \Rightarrow 6 = u+v$
$2 = \frac{1}{2}(u-v) \Rightarrow 4 = u-v$
$10 = 2u$
$u = 5$
$v = 1$

$4 = \frac{1}{2}(u+v) \Rightarrow 8 = u+v$
$3 = \frac{1}{2}(u-v) \Rightarrow 6 = u-v$
$14 = 2u$
$u = 7$
$v = 1$

$2 = \frac{1}{2}(u+v) \Rightarrow 4 = u+v$
$3 = \frac{1}{2}(u-v) \Rightarrow 6 = u-v$
$10 = 2u$
$u = 5$
$v = -1$

$3 = \frac{1}{2}(u+v) \Rightarrow 6 = u+v$
$4 = \frac{1}{2}(u-v) \quad 8 = u-v$
$14 = 2u$
$u = 7$
$v = -1$