MTH 174 Practice Exam #2 Key

1a. $$\frac{64t^3-48t^2+24t-6}{256}e^{4t}+C$$

$a=4$

b. $u=3x \rightarrow u^2=9x^2$

$du=3dx \quad x^2=\frac{1}{9}u^2$

$\frac{1}{3}du=dx \qquad a=\sqrt{2}$

$$\int\left(\frac{1}{9}\cdot\frac{1}{3}\right)u^2\sqrt{2+u^2}\,du$$

$$\frac{1}{27}\left[\frac{u}{8}(2u^2+2)\sqrt{2+u^2}-\frac{4}{8}\ln\left|u+\sqrt{2+u^2}\right|\right]+C$$

$$\frac{1}{27}\left[\frac{3x}{8}(18x^2+2)\sqrt{2+9x^2}-2\ln\left|3x+\sqrt{2+9x^2}\right|\right]+C$$

c. $u=e^x \quad du=e^x dx$

$$\int\frac{du}{1-\tan u}=\frac{1}{2}\left[u-\ln|\cos u-\sin u|\right]+C$$

$$=\frac{1}{2}\left[e^x-\ln|\cos e^x-\sin e^x|\right]+C$$

2. $u=\frac{1}{\theta} \quad du=-\frac{1}{\theta^2}d\theta$

$$-\int\cos u\,du=-\sin u+C$$

$$=-\sin\frac{1}{\theta}+C$$

3. a. $\displaystyle\int\frac{2x+1}{\sqrt{x+4}}dx$ $\qquad u=2x+1 \quad dv=(x+4)^{-1/2}dx$

$\qquad du=2dx \quad v=2(x+4)^{1/2}$

$$(2x+1)2(x+4)^{1/2}-\int 4(x+4)^{1/2}dx=$$

$$(4x+2)\sqrt{x+4}-4\cdot\frac{2}{3}(x+4)^{3/2}+C$$

3b. $\int \arctan x \, dx$ $\qquad u = \arctan x \qquad dv = dx$

$du = \frac{1}{1+x^2} dx \qquad v = x$

$$x \arctan x - \int \frac{x}{1+x^2} dx = x \arctan x - \frac{1}{2} \ln|1+x^2| + C$$

c. $\int e^{4x} \cos 2x \, dx$ $\qquad u = \cos 2x \qquad dv = e^{4x} dx$

$du = -2 \sin 2x \qquad v = \frac{1}{4} e^{4x}$

$$\frac{1}{4} e^{4x} \cos 2x + \int \frac{1}{2} e^{4x} \sin 2x \, dx$$

$u = \sin 2x \qquad dv = e^{4x} dx$

$du = 2 \cos 2x \qquad v = \frac{1}{4} e^{4x}$

$$\frac{1}{4} e^{4x} \cos 2x + \frac{1}{2} \left[\frac{1}{4} e^{4x} \sin 2x - \int \frac{1}{2} e^{4x} \cos 2x \, dx \right]$$

$$\frac{1}{4} e^{4x} \cos 2x + \frac{1}{8} e^{4x} \sin 2x - \frac{1}{4} \int e^{4x} \cos 2x \, dx = \int e^{4x} \cos 2x \, dx$$

$$+ \frac{1}{4} \int e^{4x} \cos 2x \, dx \qquad + \frac{1}{4} \int e^{4x} \cos 2x \, dx$$

$$\frac{1}{4} e^{4x} \cos 2x + \frac{1}{8} e^{4x} \sin 2x = \frac{5}{4} \int e^{4x} \cos 2x \, dx$$

$$\int e^{4x} \cos 2x \, dx = \frac{4}{5} \left[\frac{1}{4} e^{4x} \cos 2x + \frac{1}{8} e^{4x} \sin 2x \right] + C$$

$$= \frac{1}{5} e^{4x} \cos 2x + \frac{1}{10} e^{4x} \sin 2x + C$$

d.

+/-	u	dv
+	x^3	$\sin x$
−	$3x^2$	$\cos x$
+	$6x$	$-\sin x$
−	6	$-\cos x$
+	0	$\sin x$

$$x^3 \cos x + 3x^2 \sin x - 6x \cos x - 6 \sin x + C$$

e. $u = x^2 \qquad dv = x e^{x^2}$

$du = 2x \, dx \qquad v = \frac{1}{2} e^{x^2}$

$$\frac{1}{2} x^2 e^{x^2} - \int x e^{x^2} dx$$

$$= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C$$

4a. $\int \cos^4 x\,dx = \int \left[\frac{1}{2}(1+\cos 2x)\right]^2 dx = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$

$= \frac{1}{4}\int 1 + 2\cos 2x + \frac{1}{2}(1+\cos 4x)\,dx = \frac{1}{4}\int 1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\,dx$

$= \frac{1}{4}\int \frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\,dx = \frac{1}{4}\left[\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 2x\right] + C$

b. $\int \tan^5 2x\,dx = \int \tan^3 2x\,(\sec^2 2x - 1)\,dx =$

$\int \tan^3 2x \sec^2 2x\,dx - \int \tan^3 2x\,dx =$

$\int \tan^3 2x \sec^2 2x\,dx - \int \tan 2x(\sec^2 2x - 1)\,dx =$

$\int \tan^3 2x \sec^2 2x\,dx - \int \tan 2x\,(\sec^2 2x)\,dx + \int \tan 2x\,dx$

$u = \tan 2x$
$du = 2\sec^2 2x\,dx$
$\frac{1}{2}du = \sec^2 2x\,dx$

$w = 2x$
$dw = 2dx$

$\frac{1}{2}\int u^3\,du - \frac{1}{2}\int u\,du + \frac{1}{2}\int \tan w\,dw$

$\frac{1}{2}\cdot\frac{1}{4}u^4 - \frac{1}{2}\cdot\frac{1}{2}u^2 + -\frac{1}{2}\ln|\cos w| + C$

$\frac{1}{8}\tan^4 2x - \frac{1}{4}\tan^2 2x - \frac{1}{2}\ln|\cos 2x| + C$

5a. $\int \frac{1}{(1-t^2)^{5/2}}\,dt$

$t = \sin\theta$
$dt = \cos\theta\,d\theta$

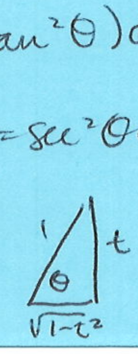
$1-t^2 = 1-\sin^2\theta = \cos^2\theta$
$(1-t^2)^{5/2} = \cos^5\theta$

$= \int \frac{\cos\theta\,d\theta}{\cos^5\theta} = \int \frac{1}{\cos^4\theta}\,d\theta = \int \sec^4\theta\,d\theta = \int \sec^2\theta\,(1+\tan^2\theta)\,d\theta$

$= \int \sec^2\theta\,d\theta + \int \sec^2\theta\tan^2\theta\,d\theta$

$u = \tan\theta \quad du = \sec^2\theta\,d\theta$
$\int u^2\,du$

$\tan\theta + \frac{1}{3}\tan^3\theta + C = \frac{t}{\sqrt{1-t^2}} + \frac{t^3}{3(1-t^2)^{3/2}} + C$



56. $\int \frac{1}{\sqrt{9+2x^2}}\,dx$ $\quad \frac{\sqrt{2}x}{3} = \tan\theta \rightarrow \sqrt{2}x = 3\tan\theta$

$2x^2 = 9\tan^2\theta$

$\frac{\sqrt{2}}{3}dx = \sec^2\theta\,d\theta$

$dx = \frac{3}{\sqrt{2}}\sec^2\theta\,d\theta$

$9+2x^2 = 9+9\tan^2\theta$

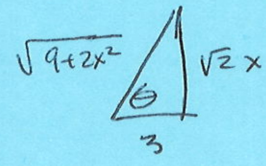
$9(1+\tan^2\theta)$

$9\sec^2\theta$

$\int \frac{1}{\cancel{3}\cancel{\sec\theta}}\cdot\frac{\cancel{3}}{\sqrt{2}}\sec^{\cancel{2}}\theta\,d\theta =$

$\sqrt{9+2x^2} = \sqrt{9\sec^2\theta} = 3\sec\theta$

$\frac{1}{\sqrt{2}}\int \sec\theta\,d\theta = \frac{1}{\sqrt{2}}\ln|\sec\theta+\tan\theta|+C$



$\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{9+2x^2}}{3}+\frac{\sqrt{2}x}{3}\right|+C$

c. $\int \frac{x^2}{\sqrt{x^2-4}}\,dx$ $\quad x = 2\sec\theta \quad x^2 = 4\sec^2\theta$

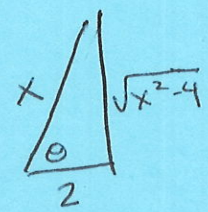
$dx = 2\sec\theta\tan\theta \quad \sqrt{x^2-4} = \sqrt{4\sec^2\theta-4} = \sqrt{4(\sec^2\theta-1)} =$

$\sqrt{4\tan^2\theta} = 2\tan\theta$

$\int \frac{4\sec^2\theta\cdot\cancel{2}\sec\theta\cancel{\tan\theta}}{\cancel{2}\cancel{\tan\theta}}\,d\theta$

$4\int \sec^3\theta\,d\theta$ $\quad u = \sec\theta \quad dv = \sec^2\theta\,d\theta$

$du = \sec\theta\tan\theta\,d\theta \quad v = \tan\theta$



$4\left[\sec\theta\tan\theta - \int \sec\theta\,\underbrace{\tan^2\theta}_{\sec^2\theta-1}\,d\theta\right]$

$4\left[\sec\theta\tan\theta - \int\sec^3\theta\,d\theta - \int\sec\theta\,d\theta\right]$

$4[\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|] - 4\int\sec^3\theta\,d\theta = \int\sec^3\theta\,d\theta$

$+4\int\sec^3\theta\,d\theta \quad 4\int\sec^3\theta\,d\theta$

$4\sec\theta\tan\theta - 4\ln|\sec\theta+\tan\theta| = 5\int\sec^3\theta\,d\theta$

$\frac{4}{5}\sec\theta\tan\theta - \frac{4}{5}\ln|\sec\theta+\tan\theta| + C = \int\sec^3\theta\,d\theta$

$= \frac{4}{5}\cdot\frac{x}{2}\cdot\frac{\sqrt{x^2-4}}{2} - \frac{4}{5}\ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right| + C = \frac{x\sqrt{x^2-4}}{5} - \frac{4}{5}\ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right| + C$

6. $$\int \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{Dx+E}{x^2+4}\, dx$$

7.
a. $$\int \frac{2}{9x^2-1}\, dx \qquad \frac{2}{9x^2-1} = \frac{2}{(3x-1)(3x+1)} = \frac{A}{3x-1} + \frac{B}{3x+1} = \frac{3Ax+A+3Bx-B}{9x^2-1}$$

$2 = 3Ax + A + 3Bx - B$

$3A + 3B = 0 \rightarrow A + B = 0$

$A = -B$

$A - B = 2$

$A + A = 2$

$2A = 2$

$A = 1 \quad B = -1$

$$\int \frac{2}{9x^2-1}\, dx = \int \frac{1}{3x-1} - \frac{1}{3x+1}\, dx = \frac{1}{3}\ln|3x-1| - \frac{1}{3}\ln|3x+1| + C$$

b. $$\int \frac{x-1}{x(x+2)(x+1)}\, dx \qquad \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+1} = \frac{A(x^2+3x+2) + Bx(x+1) + Cx(x+2)}{\sim}$$

$Ax^2 + 3Ax + 2A + Bx^2 + Bx + Cx^2 + 2Cx = x - 1$

$A + B + C = 0 \qquad -\frac{1}{2} + B + C = 0 \rightarrow B + C = \frac{1}{2}$

$3A + B + 2C = 1 \qquad -\frac{3}{2} + B + 2C = 1 \qquad B + 2C = \frac{5}{2}$

$2A = -1 \qquad -B - C = -\frac{1}{2}$

$A = -\frac{1}{2} \qquad C = 2 \qquad B + 2 = \frac{1}{2}$

$B = -\frac{3}{2}$

$$\int \frac{-1/2}{x} - \frac{3/2}{x+2} + \frac{2}{x+1}\, dx =$$

$$-\frac{1}{2}\ln|x| - \frac{3}{2}\ln|x+2| + 2\ln|x+1| + C$$

8a. improper because one limit is ∞

$u = e^x \quad du = e^x dx$

$$\lim_{b\to\infty} \int_0^b \frac{1}{e^x + e^{-x}}\, dx \cdot \frac{e^x}{e^x} = \lim_{b\to\infty} \int_0^b \frac{e^x}{e^{2x}+1}\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{u^2+1}\, du =$$

$$\lim_{b\to\infty} \arctan u \Big|_1^b = \lim_{b\to\infty} \arctan b - \arctan 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \text{ Converges}$$

8b. $\int_1^\infty \frac{1}{x\ln x}\,dx$ improper because of both ∞ and when $x=1$ function has a 0 in denominator $\Rightarrow$ not defined

$\lim_{a\to 1}\lim_{b\to\infty}\int_a^b \frac{1}{x\ln x}\,dx$ $\quad u=\ln x \rightarrow a=0$ $\quad du=\frac{1}{x}dx$

$\lim_{a\to 0}\lim_{b\to\infty}\int_a^b \frac{1}{u}\,du = \lim_{b\to\infty}\lim_{a\to 0} \ln|u| = \ln b-\ln a = \infty+\infty$

diverges

9. if ∞ or $-\infty$ is a limit, the integral is improper

if the limits are a, b, the integral is improper if

either 1) the function is not defined at a

2) the function is not defined at b

3) the function is not defined at some point on the interval (a,b).

to evaluate/analyze. determine all problem points.

Break up interval if both limits are problematic or if a problem point is inside the interval

replace problem limits w/ a dummy variable and integrate

take the limit of the result as dummy variable goes to value

if split up, both pieces must converge for the integral to converge

10. this integral represents M_x, part of the center of mass calculation for $\bar{y}$ for the region bounded by $y=-x+4$, and $y=x$ and $x=0$.

11. this integral represents the arclength of a function whose derivative is $\frac{1}{\sqrt{x}}$. if $y'=\frac{1}{\sqrt{x}}$ then $y=2\sqrt{x}+c$ on the interval [1,9].

12. $\sinh 2x = \dfrac{e^{2x}-e^{-2x}}{2}$

a.

$$2\sinh x \cosh x = \cancel{2}\left(\frac{e^{x}-e^{-x}}{\cancel{2}}\right)\left(\frac{e^{x}+e^{-x}}{2}\right) = \frac{e^{2x}-e^{-2x}}{2}$$

$\sinh 2x = 2\sinh\cosh x$

b. $$\frac{d}{dx}[\tanh x] = \frac{d}{dx}\left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right] = \frac{(e^{x}+e^{-x})(e^{x}+e^{-x})-(e^{x}-e^{-x})(e^{x}-e^{-x})}{(e^{x}+e^{-x})^2}$$

$$= \frac{e^{2x}+2+e^{-2x}-(e^{2x}-2+e^{-2x})}{(e^{x}+e^{-x})^2} = \frac{\cancel{e^{2x}}+2+\cancel{e^{-2x}}-\cancel{e^{2x}}+2-\cancel{e^{-2x}}}{(e^{x}+e^{-x})^2} =$$

$$\frac{4}{(e^{x}+e^{-x})^2} = \left(\frac{2}{e^{x}+e^{-x}}\right)^2 = \operatorname{sech}^2 x$$

Since $\cosh x = \dfrac{e^{x}+e^{-x}}{2}$

$\operatorname{sech} x = \dfrac{2}{e^{x}+e^{-x}}$

13. $f'(x) = e^{\cosh x^2}\sinh x^2 \cdot 2x$

$g'(x) = \dfrac{1}{\tanh x - \operatorname{sech} x}\cdot(\operatorname{sech}^2 x + \operatorname{sech} x \tanh x)$

$h'(x) = \cosh^3 x + 2\sinh^2 x \cosh x$

14. a. $\int \operatorname{csch}^2 x\, dx = -\coth x + C$

b. $\int \sinh x \cosh^2 x\, dx$ $u = \cosh x$, $du = \sinh x\, dx$ $\int u^2\, du = \frac{1}{3}u^3 + C$

$= \frac{1}{3}\cosh^3 x + C$

c. $\int \dfrac{\operatorname{sech}^2 x}{1+\tanh^2 x}\, dx$ $u = \tanh x$, $du = \operatorname{sech}^2 x\, dx$ $\int \dfrac{du}{1+u^2} = \arctan u + C$

$\arctan(\tanh x) + C$