MTH 291 Practice Exam #2 Key

1. a. $2y'' - y' - y = 0$

$2r^2 - r - 1 = 0$

$(2r+1)(r-1) = 0$

$r = -\frac{1}{2}, 1$

$y(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{t}$

b. $y'' - 2y' + 2y = 0$

$r^2 - 2r + 2 = 0$

$r = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$y(t) = c_1 e^t \sin t + c_2 e^t \cos t$

c. $y'' - 18y' + 81y = 0$

$r^2 - 18r + 81 = 0$

$(r-9)^2 = 0$

$r = 9$

$y(t) = c_1 e^{9t} + c_2 t e^{9t}$

2a. $Y(x) = A \sin 3x + B \cos 3x$

b. $Y(x) = Ae^x \sin x + Be^x \cos x$

c. $Y(x) = Ae^x + B$

d. $Y(t) = Ate^{-t} + t(Ct + D)$

e. $Y(t) = A + B\cos 2t + C \sin 2t$ $\qquad \cos^2 t = \frac{1}{2}(1 + \cos 2t)$

3. The natural frequency of the system is determined w/o damping. The quasifrequency is the ~~frequency~~ frequency of the exponentially decaying functions determined w/ damping included.

4. Beats occur when forcing and natural frequency functions have similar but non-identical frequencies that sometimes act constructively and sometimes destructively as they fall in and out of sync.

5. $(1-x^2)y'' - 2xy' + 2y = 0 \qquad y_1(x) = x \qquad y_2(x) = v(x)\cdot x$

$y_2' = v + xv'$

$y_2'' = 2v' + xv''$

$(1-x^2)(2v' + xv'') - 2x(v + xv') + 2xv = 0$

$2v' + xv'' - 2x^2v' - x^3v'' - \cancel{2xv} - 2x^2v' + \cancel{2xv} = 0$

$2v' + xv'' - 4x^2v' - x^3v'' = 0$

$v''(x - x^3) + v'(2 - 4x^2) = 0 \qquad u = v'$

$v''(x - x^3) = (4x^2 - 2)v' \qquad \frac{du}{dx} = v''$

$\frac{du}{u} = \frac{4x^2-2}{x-x^3}dx \qquad \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$

$x(1-x^2) = x(1-x)(1+x)$

$A - Ax^2 + Bx + Bx^2 + Cx - Cx^2 = 4x^2 - 2$

$-Ax^2 + Bx^2 - Cx^2 = 4x^2 \qquad -A + B - C = 4$

$0x = Bx + Cx \qquad B + C = 0$

$A = -2$

$2 + B - C = 4$

$B - C = 2$

$B + C = 0$

$2B = 2$

$B = 1$

$C = -1$

$\int \frac{du}{u} = \int \frac{-2}{x} + \frac{1}{1-x} - \frac{1}{1+x}\,dx$

$\ln u = -2\ln x - \ln|1-x| - \ln|1+x|$

$= \ln\left(\frac{1}{x^2}\cdot\frac{1}{1-x}\cdot\frac{1}{1+x}\right)$

$\cancel{\frac{A}{x}} + \frac{B}{x^2} + \frac{C}{1-x} + \frac{D}{1+x}$

$u = \frac{1}{x^2(1-x)(1+x)} = v'$

$Ax(1-x^2) + B(1-x^2) + Cx^2(1+x) + Dx^2(1-x)$

$Ax - Ax^3 + B - Bx^2 + Cx^2 + Cx^3 + Dx^2 - Dx^3 = 1$

$v = \int \frac{1}{x^2(1-x)(1+x)}dx$

$-\cancel{A} + C - D = 0 \qquad C - D = 0$

$-B + C + D = 0 \qquad C + D = 1$

$A = 0 \qquad 2C = 1$

$B = 1 \qquad C = \frac{1}{2}$

$D = \frac{1}{2}$

$v = \int \frac{1}{x^2} + \frac{1/2}{1-x} + \frac{\frac{1}{2}}{x+1}\,dx$

$v = -\frac{1}{x} - \frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|x+1| = -\frac{1}{x} + \ln\sqrt{\frac{x+1}{1-x}} \qquad y_2 = xv \Rightarrow$

$\boxed{y_2 = -1 + x\ln\sqrt{\frac{x+1}{1-x}}}$

6. $12 = k(1/2)$ $\quad$ $6\text{ in} = 1/2\text{ ft}$

$k = 24$

$\gamma = 3$

$12 = 32m$

$m = \frac{3}{8}$ slugs

$my'' + \gamma y' + ky = F(t)$

$\frac{3}{8}y'' + 3y' + 24y = 0$ $\quad$ $y(0) = -1$, $y'(0) = 0$

$3y'' + 24y' + 192y = 0$

$y'' + 8y' + 64y = 0$

$r^2 + 8r + 64 = 0$

$$r = \frac{-8 \pm \sqrt{64 - 4(64)}}{2} = \frac{-8 \pm 8\sqrt{3}i}{2}$$

a. underdamped $\quad$ $r = -4 \pm 4\sqrt{3}i$

b. $y(t) = c_1 e^{-4t}\sin(4\sqrt{3}t) + c_2 e^{-4t}\cos(4\sqrt{3}t)$

$y(0) = 1 = \cancel{c_1(1)(0)}^{0} + c_2(1)(1)$

$c_2 = 1$

$y'(t) = -4c_1 e^{-4t}\sin(4\sqrt{3}t) + 4\sqrt{3}c_1 e^{-4t}\cos(4\sqrt{3}t)$

$\quad -4e^{-4t}\cos(4\sqrt{3}t) - 4\sqrt{3}e^{-4t}\sin(4\sqrt{3}t)$

$y'(0) = 0 = \cancel{-4c_1(1)(0)}^{0} + 4\sqrt{3}c_1(1)(1)$

$\quad - 4(1)(1) - \cancel{4\sqrt{3}(1)(0)}^{0}$

$4\sqrt{3}c_1 - 4 = 0$

$4\sqrt{3}c_1 = 4$

$c_1 = \frac{1}{\sqrt{3}}$

$y(t) = \frac{1}{\sqrt{3}}e^{-4t}\sin(4\sqrt{3}t) + e^{-4t}\cos(4\sqrt{3}t)$

c. $T = \frac{2\pi}{\omega} = \frac{2\pi}{4\sqrt{3}}$ quasi-period

Amplitude $\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1^2} = \sqrt{\frac{1}{3}+1} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$

$\theta = \tan^{-1}\left(\frac{1}{1/\sqrt{3}}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ phase shift

d. as $t \to \infty$, $y \to 0$ oscillation decays

7. $y''+6y'+9y = 4e^{2t}+e^{-t}$

$r^2+6r+9=0$

$(r+3)^2=0$

$r=-3$

$y_1 = e^{-3t} \quad y_2 = te^{-3t}$

$W = \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & e^{-3t}-3te^{-3t} \end{vmatrix} = e^{-6t} - \cancel{3te^{-6t}} + \cancel{3te^{-6t}} = e^{-6t}$

$Y(t) = -e^{-3t}\int \frac{(4e^{2t}+e^{-t})te^{-3t}}{e^{-6t}}\,dt$

$+ te^{-3t}\int \frac{(4e^{2t}+e^{-t})e^{-3t}}{e^{-6t}}\,dt$

$= -e^{-3t}\int e^{6t}(4te^{-t}+te^{-4t})\,dt + te^{-3t}\int e^{6t}(4e^{-t}+e^{-4t})\,dt$

$= -e^{-3t}\int 4te^{5t} + te^{2t}\,dt + te^{-3t}\int 4e^{5t}+e^{2t}\,dt$

$= -e^{-3t}\left[\frac{4}{5}te^{5t} - \frac{4}{25}e^{5t} + \frac{t}{2}e^{2t} - \frac{1}{4}e^{2t}\right] + te^{-3t}\left[\frac{4}{5}e^{5t}+\frac{1}{2}e^{2t}\right]$

$= \cancel{-\frac{4}{5}te^{2t}} + \frac{4}{25}e^{2t} \cancel{- \frac{t}{2}e^{-t}} + \frac{1}{4}e^{-t} + \cancel{\frac{4}{5}te^{2t}} + \cancel{\frac{1}{2}te^{-t}} = \frac{4}{25}e^{2t} + \frac{1}{4}e^{-t}$

$y_p = c_1e^{-3t} + c_2te^{-3t} + \frac{4}{25}e^{2t} + \frac{1}{4}e^{-t}$

8. $y_1 = e^{-3t}, \; y_2 = te^{-3t}$ (see above)

$Y(t) = Ae^{2t} + Be^{-t}$

$Y'(t) = 2Ae^{2t} - Be^{-t}$

$Y''(t) = 4Ae^{2t} + Be^{-t}$

$4Ae^{2t} + Be^{-t} + 6(2Ae^{2t} - Be^{-t}) + 9(Ae^{2t}+Be^{-t}) = 4e^{2t}+e^{-t}$

$4Ae^{2t} + Be^{-t} + 12Ae^{2t} - 6Be^{-t} + 9Ae^{2t} + 9Be^{-t} = 4e^{2t}+e^{-t}$

$4A+12A+9A = 25A = 4 \qquad A = \frac{4}{25}$

$B-6B+9B = 4B = 1 \qquad B = \frac{1}{4}$

$Y(t) = \frac{4}{25}e^{2t} + \frac{1}{4}e^{-t}$

9. $Y_1(t) = At^2 + Bt + C$

$Y_1'(t) = 2At + B$

$Y_1''(t) = 2A$

$2(2A) + 3(2At + B) + At^2 + Bt + C = t^2$

$A = 1$

$6A + B = 0 \Rightarrow B = -6$

$4A + 3B + C = 0 \Rightarrow 4 - 18 + C = 0 \Rightarrow 14$

$Y_1(t) = t^2 - 6t + 14$

$Y_2(t) = D\sin t + E\cos t$

$Y_2'(t) = D\cos t - E\sin t$

$Y_2''(t) = -D\sin t - E\cos t$

$2r^2 + 3r + 1 = 0$

$(2r+1)(r+1) = 0 \quad r = -\frac{1}{2}, -1$

$y_g(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t}$

$2(-D\sin t - E\cos t) + 3(D\cos t - E\sin t) + D\sin t + E\cos t$

$(-2D - 3E + D)\sin t + (-2E + 3D + E)\cos t = 3\sin t$

$-D - 3E = 3$

$-E + 3D = 0 \Rightarrow E = 3D$

$-D - 3(3D) = 3$

$-10D = 3$

$D = -\frac{3}{10}$

$E = \frac{-9}{10}$

$Y_2(t) = \frac{-3}{10}\sin t - \frac{9}{10}\cos t$

$y(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t} + t^2 - 6t + 14 - \frac{3}{10}\sin t - \frac{9}{10}\cos t$

10. $y'' - 2y' + y = 0$

$r^2 - 2r + 1$

$(r-1)^2 = 0$

$y_1 = e^t \quad y_2 = te^t$

$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + \cancel{te^{2t}} - \cancel{te^{2t}} = e^{2t}$

$Y(t) = -e^t \int \frac{\cancel{e^t}}{1+t^2} \cdot \frac{t\cancel{e^t}}{\cancel{e^{2t}}}\, dt + te^t \int \frac{\cancel{e^t}}{1+t^2} \frac{\cancel{e^t}}{\cancel{e^{2t}}}\, dt$

$= -e^t \int \frac{t}{1+t^2}\, dt + te^t \int \frac{1}{1+t^2}\, dt$

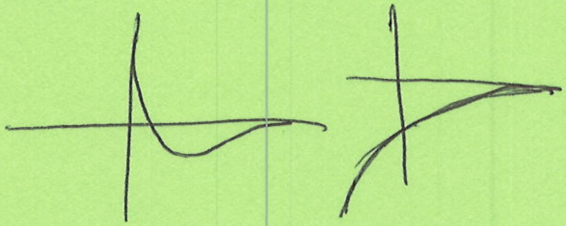
$= -\frac{1}{2}e^t \ln|1+t^2| + te^t \arctan t$

$y_p(t) = c_1 e^t + c_2 te^t - \frac{1}{2}e^t \ln|1+t^2| + te^t \arctan t$

11. a. resonance $t \sin t$ term

b. beats frequencies are 6, 7

12. overdamped has 2 distinct real, negative roots



Crosses axis a maximum of one time (possibly none)

13. a. transient e^{-t} terms underdamped

Steady state are $5\cos 4t + 4\sin 4t$

no resonance or beats

b. transient first 2 terms e^{-rt} overdamped

Steady state $\sin 3t$

no resonance or beats

c. all steady state

undamped

resonance

14. $W = e^{-\int 2x\,dx} = e^{-x^2}$

15. $W = \begin{vmatrix} t^2 & t^2\ln t \\ 2t & 2t\ln t + t \end{vmatrix} = \cancel{2t^3\ln t} + t^3 - \cancel{2t^3\ln t} = t^3$

16. $\int_0^\infty e^{-st}(1+\cosh 5t)\,dt = \int_0^\infty e^{-st} + e^{-st}\left(\frac{e^{5t}+e^{-5t}}{2}\right) dt$

$= \int_0^\infty e^{-st} + \frac{1}{2}e^{-t(s-5)} + \frac{1}{2}e^{-t(s+5)}\,dt$

$-\frac{1}{s}e^{-st} - \frac{1}{2(s-5)}e^{-t(s-5)} - \frac{1}{2(s+5)}e^{-t(s+5)}\Big|_0^\infty = 0 + \frac{1}{s} + \frac{1}{2(s-5)} + \frac{1}{2(s+5)}$

$= \frac{1}{s} + \frac{s+\cancel{5}+s-\cancel{5}}{2(s+5)(s-5)} = \frac{1}{s} + \frac{\cancel{2}s}{\cancel{2}(s^2-25)} = \frac{1}{s} + \frac{s}{s^2-25}$

17a. $\mathcal{L}\{(1+t^2)\} = \frac{1}{s} + \frac{2!}{s^{2+1}} = \frac{1}{s} + \frac{2}{s^3}$

b. $\mathcal{L}\{te^t\} = \frac{1!}{(s-1)^{1+1}} = \frac{1}{(s-1)^2}$

c. $\mathcal{L}\{e^{-2t}\sin 3\pi t\} = \frac{s+2}{(s+2)^2+9\pi^2}$

d. $\mathcal{L}^{-1}\{\frac{1}{2} - \frac{2}{s^5}\} = \frac{1}{2}\delta(t) - \frac{1}{12}t^4$

e. $\mathcal{L}^{-1}\{\frac{9-17s}{s^2+81}\} = \mathcal{L}^{-1}\{\frac{9}{s^2+81}\} - 17\mathcal{L}^{-1}\{\frac{s}{s^2+81}\} = \sin 9t - 17\cos 9t$

f. $\mathcal{L}^{-1}\{\frac{1}{s(s^2+4)}\} = \mathcal{L}^{-1}\{\frac{1}{s}\cdot\frac{1}{s^2+4}\} = \int_0^t 1\cdot\frac{1}{2}\cdot\sin 2(t-\tau)\,d\tau$

g. $\mathcal{L}^{-1}\{\frac{1}{s^2(s^2-1)}\} = \mathcal{L}^{-1}\{\frac{1}{s^2}\cdot\frac{1}{s^2-1}\} = \int_0^t t\cdot\sinh(t-\tau)\,d\tau$

18. $y'' + 4y' + 8y = e^{-t}$ $\quad y(0)=0,\ y'(0)=1$

$s^2Y(s) - \cancel{s(0)} - 1 + 4[sY(s) - 0] + 8Y(s) = \frac{1}{s+1}$

$s^2Y(s) - 1 + 4sY(s) + 8Y(s) = \frac{1}{s+1}$

$Y(s)(s^2+4s+8) = \frac{1}{s+1} + 1 = \frac{1+s+1}{s+1} = \frac{s+2}{s+1}$

$Y(s) = \frac{s+2}{(s+1)(s^2+4s+8)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+8} \rightarrow = s^2+4s+4+4$

$(s+2)^2+4$

$As^2 + 4As + 8A + Bs^2 + Cs + Bs + C = s+2$

$A + B = 0$ $\quad A = 1/5$

$4A + B + C = 1$ $\quad B = -1/5$

$8A + C = 2$ $\quad C = 2/5$

$-\frac{1}{5}[(s+2)-2] = -\frac{1}{5}(s+2) + \frac{2}{5}$

$Y(s) = \frac{1/5}{s+1} + \frac{-1/5\,s}{(s+2)^2+4} + \frac{2/5}{(s+2)^2+4}$

$= \frac{1/5}{s+1} + \frac{-1/5(s+2)}{(s+2)^2+4} + \frac{4/5}{(s+2)^2+4}$

$y(t) = \frac{1}{5}e^{-t} - 1/5\,e^{-2t}\cos 2t + 4/5\,e^{-2t}\sin 2t$

19. $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ the specifics depend on $f(t)$
practice w/ some of your choice
compare w/ table

20. $9.8(1/10) = k(.05)$ $\quad k = 19.6$ watch units SI gravity in kg & meters

a.

$y(0) = 0$

$y'(0) = -\frac{1}{10}$

$\frac{1}{10}y'' + 19.6y = 0$

$y'' + 196y = 0$

$r^2 + 196 = 0$

$r = \pm 14i$

$y(t) = c_1 \sin 14t + c_2 \cos 14t$

$y(0) = \cancel{c_1(0)} + c_2(1) = 0$

$c_2 = 0$

$y(t) = c_1 \sin 14t$

$y'(t) = 14c_1 \cos 14t$

$y'(0) = -\frac{1}{10} = 14c_1(1)$

$c_1 = -\frac{1}{140}$

$y(t) = -\frac{1}{140} \sin 14t$

b. $14t = \pi \Rightarrow t = \frac{\pi}{14} \approx .2244$ seconds

c. $P = \frac{2\pi}{14} = \frac{\pi}{7}$ period

amplitude $= \frac{1}{140}$

phase shift $= 0$