Exam #3
Calc III
Summer 2021

KEY

Instructions: Show all work. Use exact answers unless specifically asked to round. You may check your answers in the calculator, but you must show work to get full credit. Incorrect answers with no work will receive no credit. Be sure to complete all the requested elements of each problem.

1. For the curve $\vec{r}(t) = sin^3 t\hat{\imath} - cos^3 t\hat{\jmath}$, find the unit tangent vector, the unit normal vector and the binormal vector. (14 points)

$$r'(t) = 3\sin^2 t\cos t\,\hat{\imath} + 3\cos^2 t\sin t\,\hat{\jmath} = 3\sin t\cos t\,(\sin t\,\hat{\imath} + \cos t\,\hat{\jmath})$$

$$\|r'(t)\| = 3\sin t\cos t\sqrt{\sin^2 t + \cos^2 t} = 3\sin t\cos t$$

$$T(t) = \frac{3\sin^2 t\cos t\,\hat{\imath} + 3\cos^2 t\sin t\,\hat{\jmath}}{3\sin t\cos t} = \boxed{\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}}$$

$$T'(t) = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath} \qquad \|T'(t)\| = 1$$

$$\boxed{N(t) = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath}}$$

$$B(t) = T(t) \times N(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin t & \cos t & 0 \\ \cos t & -\sin t & 0 \end{vmatrix} = (-\sin^2 t - \cos^2 t)\,\hat{k}$$

$$= \boxed{-\hat{k}}$$

or $\langle 0, 0, -1 \rangle$

2. Find the directional derivative of the function $f(x, y) = x^2 e^{x+y}$ at the point (1,-1,1) in the direction $\vec{u} = 2\hat{\imath} - 3\hat{\jmath}$. (8 points)

$$\nabla f = (2xe^{x+y} + x^2 e^{x+y})\hat{\imath} + (x^2 e^{x+y})\hat{\jmath}$$

$$\nabla f(1,-1) = (2+1)\hat{\imath} + (1)\hat{\jmath} = \langle 3, 1 \rangle$$

$$\hat{u} = \frac{2}{\sqrt{13}}\hat{\imath} - \frac{3}{\sqrt{13}}\hat{\jmath} = \left\langle \frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}} \right\rangle$$

$$D_u f = \nabla f \cdot \hat{u} = \langle 3, 1 \rangle \cdot \left\langle \frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}} \right\rangle = \frac{6}{\sqrt{13}} - \frac{3}{\sqrt{13}} = \frac{3}{\sqrt{13}}$$

3. Find the equation of the tangent plane for the function $g(x, y) = y\ln|x| - x^2$ at the point (1,6,-1). Then find the equation of the line that is normal to the surface at the same point. (12 points)

$F(x,y,z) = y\ln|x| - x^2 - z \qquad \nabla F = \langle y/x - 2x, \ln|x|, -1 \rangle$

$\nabla F(1,6,-1) = \langle 6-2, 0, -1 \rangle = \langle 4, 0, -1 \rangle$

tangent plane $\quad 4(x-1) + 0(y-6) - 1(z+1) = 0$

$4(x-1) - (z+1) = 0$

normal line $\quad r(t) = (4t+1)\hat{i} + 6\hat{j} + (-t-1)\hat{k}$

4. Find the equation of the tangent plane for the parametric surface $\vec{r}(u, v) = 2\cos u \sin v\,\hat{\imath} + 2\sin u \sin v\,\hat{\jmath} + 2\cos v\,\hat{k}$ at the point $(\sqrt{2}, 0, \sqrt{2})$. Then find the equation for the normal vector to the function at any point. (12 points)

$u = 0 \qquad 2(1)\sqrt{2}/2 = \sqrt{2}$

$v = \pi/4 \qquad 2(0)(\frac{\sqrt{2}}{2}) = 0$

$2(\sqrt{2}/2) = \sqrt{2}$

$r_u = -2\sin u \sin v\,\hat{i} + 2\cos u \sin v\,\hat{j} + 0k$

$r_v = 2\cos u \cos v\,\hat{i} + 2\sin u \cos v\,\hat{j} - 2\sin v\,\hat{k}$

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u \sin v & 2\cos u \sin v & 0 \\ 2\cos u \cos v & 2\sin u \cos v & -2\sin v \end{vmatrix} =$$

$$(4\sin v \cos u)\hat{i} - (4\sin^2 v \sin u)\hat{j} + (\underbrace{-4\sin^2 u \cos v \sin v - 4\cos^2 u \cos v \sin v}_{=1})\hat{k}$$

$N = -4[\sin^2 v \cos u\,\hat{i} + \sin^2 v \sin u\,\hat{j} + \cos v \sin v\,\hat{k}]$

$N = -4\langle \frac{1}{2}(1), 0, \frac{1}{2} \rangle = \langle -2, 0, -2 \rangle$

tangent plane: $\quad -2(x-\sqrt{2}) + 0(y-0) - 2(z-\sqrt{2}) = 0$ or

$-2(x-\sqrt{2}) - 2(z-\sqrt{2}) = 0$

5. Find the length of the parametric curve $\vec{r}(t) = t\hat{\imath} + \ln|cos\, t|\,\hat{\jmath}$. (8 points)

$r'(t) = 1\hat{\imath} - \tan t\hat{\jmath}$

$\|r'(t)\| = \sqrt{1+\tan^2 t} = \sqrt{\sec^2 t} = |\sec t|$

$S = \int_a^b |\sec t|\, dt$ = defined for a, b between $-\pi/2 < t < \pi/2$

6. Find the curvature of the function $\vec{r}(t) = t\sin t\,\hat{\imath} - \cos t\,\hat{\jmath} + t^2\hat{k}$. Use the formula $\kappa = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3}$. Note any values of the parameter where the curvature is zero. (12 points)

$r'(t) = (\sin t + t\cos t)\hat{\imath} + \sin t\hat{\jmath} + 2t\hat{k}$

$r''(t) = (2\cos t - t\sin t)\hat{\imath} + \cos t\hat{\jmath} + 2\hat{k}$

$\|r'(t)\| = \sqrt{(\sin t + t\cos t)^2 + \sin^2 t + 4t^2} =$

$\sqrt{\sin^2 t + 2t\cos t\sin t + t^2\cos^2 t + \sin^2 t + 4t^2} =$

$\sqrt{2\sin^2 t + 2t\cos t\sin t + t^2\cos^2 t + 4t^2}$

$r' \times r'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin t + t\cos t & \sin t & 2t \\ 2\cos t - t\sin t & \cos t & 2 \end{vmatrix} = (2\sin t - 2t\cos t)\hat{\imath} - (2\sin t + 2t\cos t - 4t\cos t + 2t^2\sin t)\hat{\jmath} + (\sin t\cos t + t\cos^2 t - 2\sin t\cos t + t\sin^2 t)\hat{k}$

$= (2\sin t + 2t\cos t)\hat{\imath} + (2t\cos t - 2\sin t - 2t^2\sin t)\hat{\jmath} + (t - \sin t\cos t)\hat{k} \longrightarrow$

7. Find the surface area of the surface $x^2 + y^2 - z^2 = 1$ above the xy-plane and below the line $z = 4$. (10 points)

$\sqrt{x^2+y^2-1} = \sqrt{z^2} \rightarrow z = \sqrt{x^2+y^2-1}$

$x^2+y^2 = 1+z^2 \Rightarrow 17$

$x^2+y^2 = 1+z^2 \rightarrow 1$

$z=0$

$\nabla F = \left\langle \frac{x}{\sqrt{x^2+y^2-1}}, \frac{y}{\sqrt{x^2+y^2-1}}, -1 \right\rangle$

$\iint_R \sqrt{1 + \frac{x^2}{x^2+y^2-1} + \frac{y^2}{x^2+y^2-1}}\, dA = \iint_R \sqrt{\frac{x^2+y^2-1+x^2+y^2}{x^2+y^2-1}}\, dA$

$\int_0^{2\pi}\int_1^{\sqrt{17}} \sqrt{\frac{2r^2-1}{r^2-1}}\, r\,dr\,d\theta \Rightarrow$ Calculate r integral numerically

$\approx 2\pi(5.144608163) \approx 32.32452642$

6. con'td

$$\|r' \times r''\| = \sqrt{(2\sin t - 2t\cos t)^2 + (2t\cos t - 2\sin t - 2t^2\sin t)^2 + (t - \sin t\cos t)^2}$$

$$= \sqrt{4\sin^2 t - 8t\sin t\cos t + 4t^2\cos^2 t + 4t^2\cos^2 t + 4\sin^2 t + 4t^4\sin^2 t - 8t\sin t\cos t - 8t^3\sin t\cos t + 8t^2\sin^2 t + t^2 - 2t\sin t\cos t + \cos^2 t\sin^2 t}$$

$$= \sqrt{8\sin^2 t - 18t\sin t\cos t + \underbrace{8t^2\cos^2 t + 4t^4\sin^2 t - 8t^3\sin t\cos t + 8t^2\sin^2 t}_{= 8t^2} + t^2 + \cos^2 t\sin t}$$

$$= \sqrt{\sin^2 t\,(8 + 4t^4) - \sin t\cos t\,(18t + 8t^3) + \cos^2 t\sin^2 t + 9t^2}$$

$$K = \frac{\sqrt{\sin^2 t\,(8 + 4t^4) - \sin t\cos t\,(18t + 8t^3) + \cos^2 t\sin^2 t + 9t^2}}{(2\sin^2 t + 2t\cos t\sin t + t^2\cos^2 t + 4t^2)^{3/2}}$$

to find places where the curve may have zero curvature, you could graph the function. The numerator will be zero at $t=0$, but so will the denominator

there are no other values where the expression is obviously equal to zero.

8. Find the value of $\int_C \vec{F}\cdot d\vec{r}$ for the vector field $\vec{F}(x,y,z) = \frac{1}{3}x^2z\hat{\imath} + \frac{1}{2}y\hat{\jmath} + \frac{1}{9}x^3\hat{k}$ from the point (2,-1,0) to the point (1,1,3). If the field is conservative, use the Fundamental Theorem of line integrals. If it is not, find the work done on the straight-line path between the two points. (12 points)

$$\int \tfrac{1}{3}x^2 z\,dx = \tfrac{1}{9}x^3 z + g(y,z) \qquad \int \tfrac{1}{2}y\,dy = \tfrac{1}{4}y^2 + h(x,z)$$

$$\int \tfrac{1}{9}x^3\,dz = \tfrac{1}{9}x^3 z + j(x,y)$$

$$f(x,y,z) = \tfrac{1}{9}x^3 z + \tfrac{1}{4}y^2$$

potential function exists
field is conservative
apply fundamental theorem
of line integrals

$$\int_C F\cdot dr = f(1,1,3) - f(2,-1,0) = \tfrac{1}{9}(1)^3(3) + \tfrac{1}{4}(1)^2 - \tfrac{1}{9}(2)^3(0) - \tfrac{1}{4}(-1)^2$$

$$= \tfrac{1}{3} + \tfrac{1}{4} - 0 - \tfrac{1}{4} = \tfrac{1}{3}$$

9. Find the value of the line integral $\int_C (2x-y)dx + (y+5x)dy$ around the area bounded by the rectangle with vertices (0,0), (2,0), (2,5), (0,5), traversed from (0,0) counterclockwise. [Hint: Use Green's Theorem.] (10 points)

$$\frac{\partial N}{\partial x} = 5 \qquad \frac{\partial M}{\partial y} = -1 \qquad \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA =$$

$$\int_0^2\int_0^5 5-(-1)\,dy\,dx$$

$$= 6\int_0^2\int_0^5 dy\,dx = 6(2)(5) = 60$$

10. Compute the value of the surface integral $\int\int_S g(x,y,z)dS$ for $g = xy$ over the triangle $x + y + z = 1, x, y, z \geq 0$. (10 points)

$$x+y=1 \qquad \nabla F = \langle 1,1,1\rangle$$
$$y = -x+1 \qquad \|\nabla F\| = \sqrt{3}$$

$$\int_0^1\int_0^{-x+1} \sqrt{3}\,xy\,dy\,dx = \frac{\sqrt{3}}{2}\int_0^1 xy^2\Big|_0^{-x+1} dx = \frac{\sqrt{3}}{2}\int_0^1 x^3 - 2x^2 + x\,dx$$

$$= \frac{\sqrt{3}}{2}\left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 + \tfrac{1}{2}x^2\right]_0^1 = \frac{\sqrt{3}}{2}\left[\tfrac{1}{4} - \tfrac{2}{3} + \tfrac{1}{2}\right] = \left[\tfrac{1}{12}\right]\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{24}$$

11. Find the value of the surface integral $\int\int_S \|\vec{r_u} \times \vec{r_v}\| dS$ for $\vec{r}(u,v) = uv\hat{\imath} + (u+v)\hat{\jmath} + (u-v)\hat{k}$, for $0 \le u \le 3, 1 \le v \le 5$. (12 points)

$$r_u \times r_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} = (-1-1)\hat{\imath} - (-v-u)\hat{\jmath} + (v-u)\hat{k}$$

$$= (-2)\hat{\imath} + (u+v)\hat{\jmath} + (v-u)\hat{k}$$

$$\|r_u \times r_v\| = \sqrt{4 + u^2 + v^2 + 2uv + u^2 + v^2 - 2uv} = \sqrt{4 + 2u^2 + 2v^2}$$

$$\iint_R \|r_u \times r_v\| \, dS = \iint \left(\sqrt{4 + 2u^2 + 2v^2}\right)^2 dA = \int_0^3 \int_1^5 4 + 2u^2 + 2v^2 \, dv\, du$$

$$= \int_0^3 16 + 8u^2 + \frac{248}{3} \, du = \frac{296}{3}(3) + 72 = 368$$

12. Find the flux $\int\int_S \vec{F} \cdot \vec{N} dS$ by the Divergence Theorem for $\vec{F}(x,y,z) = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$ on the unit sphere. (10 points)

$$\operatorname{div} F = \nabla \cdot F = 2x + 2y + 2z \qquad \iint_S F \cdot N \, dS = \iiint_Q \operatorname{div} F \, dV$$

$$\iiint_Q 2x + 2y + 2z \, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 (2\rho \cos\theta \sin\varphi + 2\rho \sin\theta \sin\varphi + 2\rho\cos\varphi)\rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= 2\int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^3(\cos\theta \sin^2\varphi + \sin\theta \sin^2\varphi + \cos\varphi \sin\varphi)\, d\rho\, d\theta\, d\varphi =$$

$$\frac{1}{2}\int_0^{\pi}\int_0^{2\pi} \cos\theta \sin^2\varphi + \sin\theta \sin^2\varphi + \cos\varphi \sin\varphi \, d\theta\, d\varphi =$$

$$\frac{1}{2}\int_0^{\pi} \Big[\underbrace{\sin\theta \sin^2\varphi}_{0} + \underbrace{-\cos\theta \sin^2\varphi}_{\substack{-(1-1) \\ 0}}\Big]_0^{2\pi} + 2\pi \cos\varphi \sin\varphi \, d\varphi =$$

$$\frac{2\pi}{2}\int_0^{\pi} \cos\varphi \sin\varphi \, d\varphi = \pi\left[\frac{1}{2}\sin^2\varphi\right]_0^{\pi} = 0$$

13. Use Stokes' Theorem to compute $\int_C \vec{F} \cdot d\vec{r} = \int\int_S (\nabla \times \vec{F}) \cdot \vec{N} dS$ for $\vec{F}(x, y, z) = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$ over the top half of the unit sphere. (13 points)

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & z^2 \end{vmatrix} = 0\hat{i} - 0\hat{j} - 0\hat{k}$$

$$\iint_S (\nabla \times F) \cdot N \, dS = \iint_S 0 \cdot N \, dS = 0$$

Since F is conservative and the curve is closed, this should be zero.