5/24/2021

On the archive site, some of the handouts have answer keys posted for the practice problems at the end of the handout. 3d point-plotting app at https://technology.cpm.org/general/3dgraph/ that let's you plot by coordinate points.

Line integrals: $ds = ||r'(t)||dt$, in two dimensions $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$, in three-dimensions $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

Differentials

In single-variable functions, the differential was $dy = f'(x)dx$, so we can approximate $\Delta y \approx f'(x)\Delta x$

The new y-value $y + \Delta y \approx f(x) + f'(x)\Delta x$

Two-variable function $z = f(x, y)$. Total differential $dz = f_x(x, y)dx + f_y(x, y)dy$

Our approximation then is $\Delta z \approx f_x(x, y)\Delta x + f_y(x, y)\Delta y$

$$z + \Delta z \approx f(x, y) + f_x(x, y)\Delta x + f_y(x, y)\Delta y$$

Three-variable function $w = f(x, y, z)$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

1) Find an expression for the differential at a given point
2) Estimate the value of a function near another "nice" point

Find the total differential for $z = \arctan(xy^2)$ at the point $\left(1,1,\frac{\pi}{4}\right)$.

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

$$dz = \left(\frac{y^2}{1 + (xy^2)^2}\right)dx + \left(\frac{2xy}{1 + (xy^2)^2}\right)dy$$

$$dz\left(1,1,\frac{\pi}{4}\right) = \left(\frac{1}{2}\right)dx + \left(\frac{2}{2}\right)dy$$

Estimate the value of $z = 5x^2 + y^2$ at the $(1.05, 2.1)$ using differentials.

The point we will estimate from is (1,2). $\Delta x = 0.05, \Delta y = 0.1$.

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

$$\Delta z \approx (10x)\Delta x + (2y)\Delta y = (10)(0.05) + (4)(0.1) = 0.5 + 0.4 = 0.9$$

$$z(1.05,2.1) \approx 5(1)^2 + (2)^2 + 0.9 = 9.9$$

Compare to the real value to check: $5(1.05)^2 + (2.1)^2 = 9.9225$

Del Notation

∂ = lower case "del"

Capital "del" is like an upside-down Delta: $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ This is a vector of derivative operators.

Combine the ∇ operator with a function $f(x, y, z)$

$$\nabla f = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle f_x, f_y, f_z \rangle$$

The resulting vectors is a vector that contains all the partial derivatives (first partials) for the function f.

∇f (del-f) = gradient of the function = grad(f). Grad(f) is a vector field.

$$f(x, y, z) = x^2 - y^2 + 6xyz + z^2$$

$$\nabla f = \langle 2x + 6yz, -2y + 6xz, 6xy + 2z \rangle$$

$\nabla \cdot F$ = div(F) is called the divergence of a vector field. Not operating on a function anymore. $F(x, y, z)$ is a vector field. The result of this operation is a function (scalar).

$$F(x, y, z) = \langle x^2 y, y^3 z, xyz \rangle$$

$$\nabla \cdot F = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle x^2 y, y^3 z, xyz \rangle = \frac{\partial}{\partial x}[x^2 y] + \frac{\partial}{\partial y}[y^3 z] + \frac{\partial}{\partial z}[xyz] =$$
$$2xy + 3y^2 z + xy = 3xy + 3y^2 z$$

$\nabla \times$ F = curl(F) is called the curl of a vector field.

$$\nabla \times \mathrm{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & y^3 z & xyz \end{vmatrix} =$$

$$\left(\frac{\partial}{\partial y}[xyz] - \frac{\partial}{\partial z}[y^3 z]\right)\hat{\imath} - \left(\frac{\partial}{\partial x}[xyz] - \frac{\partial}{\partial z}[x^2 y]\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}[y^3 z] - \frac{\partial}{\partial y}[x^2 y]\right)\hat{k} =$$

$$(xz - y^3)\hat{\imath} - (yz - 0)\hat{\jmath} + (0 - x^2)\hat{k}$$

$$(xz - y^3)\hat{\imath} - yz\hat{\jmath} - x^2\hat{k}$$

A vector field is the result.

The Laplacian is dotting the del-vector with the gradient.

$$\nabla \cdot (\nabla f) = \nabla^2 f = f_{xx} + f_{yy} + f_{zz}$$

$$f(x, y, z) = x^2 - y^2 + 6xyz + z^2$$

$$\nabla f = \langle 2x + 6yz, -2y + 6xz, 6xy + 2z \rangle$$

Find the Laplacian:

$$\nabla \cdot \langle 2x + 6yz, -2y + 6xz, 6xy + 2z \rangle =$$
$$2 + (-2) + 2 = 2$$

If a vector is equivalent to the gradient of some function, then the function is referred to as a potential function.

If you take the curl of a gradient function, then the curl is zero. A vector field with a zero curl everywhere is called conservative.