6/15/2021

Divergence Theorem/Stokes' Theorem (16.9, 16.8)

Last time: we talked about surface integrals (generally), and the field version represents a flow through the surface. We mostly talked about flow through a single surface, generally not a close surface.

In this section, we are going to talk about flow through a closed surface. We don't care about the properties of the field, but we do care that the surface is closed, so the surface represents the closure of some volume of space.

We are going to convert the flow through the closed surface into a volume integral.

We are going to assume a positive orientation: the flow is outward relative to the surface. Because we are not doing the individual surfaces by hand, we don't have to calculate the orientation of the surfaces when applying the Divergence theorem.

Example where Divergence Theorem can be applied:

$$F(x, y, z) = \langle 3x, xy, 2xz \rangle$$

The region (volume) is the cube bounded by the planes: $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

Integral #1: plane $x = 0$. Step1: plug x=0 into the field. Step 2: if the field is non-zero, then integrate in the remaining variables (in this case the surface is square with bounds $y = 0, y = 1, z = 0, z = 1$. Step3: Find your normal vector pointing outward: $n = \langle -1,0,0 \rangle$.
Integral #2: plane $x = 1$., Step 1: plug in x=1 into the field. Step 2: set up with same limits as before. Step 3: the normal vector is $n = \langle 1,0,0 \rangle$. Here, you'd integrate 3, over y and z (a square).
Integral #3: plane y=0. Step 1: plug in y=0. Step 2: set up integral to integrate over x and z, $= 0, x = 1, z = 0, z = 1$. Step 3: dot the normal vector $n = \langle 0, -1,0 \rangle$.
Integral #4: plane y=1. Step 1: plug in y=1 into field. Step 2: set up integral to integrate over the square in x and z. Step 3: dot the field with the normal vector $n = \langle 0,1,0 \rangle$.
Integral #5: plane z=0. Step 1: plug in z=0 into field. Step 2: set up integral in x and y. Step 3: the normal vector $n = \langle 0,0, -1 \rangle$, and dot with the field.
Integral #6: plane z=1. Step 1: plug in z=1 into the field. Step 2: step up the integral to integrate over the square in x and y. Step 3: dot the field with the normal vector $n = \langle 0,0,1 \rangle$.
Finally, add up the results of all the integrals…. I think $3 + \frac{1}{2} + 1 = \frac{9}{2}$.

Divergence Theorem:

$$\iint_S \vec{F} \cdot d\vec{S} = \oiint \vec{F} \cdot d\vec{S} = \iiint_V div\, \vec{F} dV = \iiint_V \nabla \cdot \vec{F} dV$$

If the flow is incompressible, then the integral will be zero (the same amount is flowing in as flowing out).
If the result is positive, then there is source inside the volume (enclosed surface): more coming out than is going in.
If the result is negative, then there is a sink inside the volume (enclose surface): more going in than is coming out.

$$F(x,y,z) = \langle 3x, xy, 2xz \rangle$$

The region (volume) is the cube bounded by the planes: $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

$$\iint_S F \cdot dS = \iint_{S_1} F \cdot dS_1 + \iint_{S_2} F \cdot dS_2 + \iint_{S_3} F \cdot dS_3 + \iint_{S_4} F \cdot dS_4 + \iint_{S_5} F \cdot dS_5 + \iint_{S_6} F \cdot dS_6$$

$$= \iiint_V \nabla \cdot F dV = \iiint_V \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle 3x, xy, 2xz \rangle dV = \iiint_V (3 + x + 2x) dV$$

$$= \int_0^1 \int_0^1 \int_0^1 (3 + 3x)\, dzdydx$$

$$= \int_0^1 \int_0^1 (3 + 3x) z|_0^1 dydx = \int_0^1 \int_0^1 (3 + 3x) dydx = \int_0^1 (3 + 3x) dx = 3x + \frac{3}{2}x^2\Big|_0^1 = 3 + \frac{3}{2} = \frac{9}{2}$$

Example.

$$F(x,y,z) = \langle \cos z + xy^2, xe^{-z}, \sin y + x^2 z \rangle$$

S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Old way: two integrals: one on the paraboloid, and on the plane $z = 4$. The normal vector for the top plane would be $n = \langle 0,0,1 \rangle$. Dot that with the field where z=4 is plugged in. Integrating in $x$ and $y$, but intersection with the plane is a circle of radius 2: $4 = x^2 + y^2$.

$$\int_0^{2\pi} \int_0^2 (\sin(r \sin\theta) + 4r^2 \cos^2\theta) r dr d\theta + other\ integral$$

Other integral: $G(x,y,z) = x^2 + y^2 - z, \nabla G = \langle 2x, 2y, -1 \rangle$

$$\iint_S \langle \cos(x^2 + y^2) + xy^2, xe^{-(x^2+y^2)}, \sin y + x^2(x^2 + y^2) \rangle \cdot \langle 2x, 2y, -1 \rangle dA$$

This is awful. The Divergence Theorem not only reduces the number of integrals to compute, but the functions are easier to set up, and because we are taking derivatives, the functions we are integrating often get easier.

The new way:

$$div\ F = \nabla \cdot F = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle \cos z + xy^2, xe^{-z}, \sin y + x^2 z \rangle = y^2 + 0 + x^2 = x^2 + y^2 = r^2$$

$$\int_0^{2\pi} \int_0^2 \int_{r^2}^4 (r^2) r dz dr d\theta = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 (r^3) dz dr d\theta = \int_0^{2\pi} \int_0^2 r^2(4 - r^2) dr d\theta =$$

$$\int_0^{2\pi} \int_0^2 (4r^2 - r^4) dr d\theta = \int_0^{2\pi} \frac{4}{3} r^3 - \frac{1}{5} r^5 \Big|_0^1 d\theta = \left(\frac{4}{3} - \frac{1}{5}\right) 2\pi = \frac{34\pi}{15}$$

Stokes' Theorem (16.8)

A way of looking at line integrals through the lens of surfaces. Curves in space can be defined by the intersection of surfaces.

The purpose of this theorem is to convert a line integral in one dimension into a two-dimensional surface integral.

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

Assumes the same positive orientation around the curve (counterclockwise/area on the left) that we used with Green's Theorem.

Example.

$$F(x,y,z) = \langle x^2z^2, y^2z^2, xyz \rangle$$

S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$.
(what is the curve of intersection: $r(t) = \langle 2\cos t, 2\sin t, 4 \rangle$)
Oriented upward (z is positive).

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z^2 & x^2y^2 & xyz \end{vmatrix} = (xz - 0)i - (yz - 2x^2z)j + (2xy^2 - 0)k =$$

$$\langle xz, 2x^2z - yz, 2xy^2 \rangle$$

Normal vector to the surface:

$$G(x,y,z) = z - x^2 - y^2$$
$$\nabla G = \langle -2x, -2y, 1 \rangle$$

$$\iint_R \langle xz, 2x^2z - yz, 2xy^2 \rangle \cdot \langle -2x, -2y, 1 \rangle \, dA \iint_R -2x^2z - 2x^2yz + 2y^2z + 2xy^2 dA$$

$$\iint_R (-2x^2(x^2+y^2) - 2x^2y(x^2+y^2) + 2y^2(x^2+y^2) + 2xy^2) dA$$

$$\int_0^{2\pi}\int_0^2 (-2r^4\cos^2\theta - 2r^5\cos^2\theta\sin\theta + 2r^4\sin^2\theta + 2r^3\cos\theta\sin^2\theta)\, r dr d\theta$$

$$= \int_0^{2\pi}\int_0^2 -2r^5\cos 2\theta - 2r^6\cos^2\theta\sin\theta + 2r^4\cos\theta\sin^2\theta \, dr d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}r^6\cos 2\theta - \frac{2}{7}r^7\cos^2\theta\sin\theta + \frac{2}{5}r^5\cos\theta\sin^2\theta \Big|_0^2 \, d\theta =$$

$$\int_0^{2\pi} -\frac{64}{3}\cos 2\theta - \frac{256}{7}\cos^2\theta\sin\theta + \frac{64}{5}\cos\theta\sin^2\theta\, d\theta =$$

$$-\frac{32}{3}\sin 2\theta + \frac{256}{21}\cos^3\theta + \frac{64}{15}\sin^3\theta\Big|_0^{2\pi} = 0$$

Unless the problem says specifically to apply Stokes' theorem, it is okay to apply Green's Theorem to an equivalent curve.

$$z = 4 - x^2 - y^2, z = 0$$

Surface in the xy-plane had a normal vector of $n = \langle 0,0,\pm 1\rangle$.
Choose, when the problem is less specific, your surface wisely: pick the one that needs the least amount of math.

Next time: we'll do more Stokes' examples, and then review for the Exam #3 (Thursday).