MTH 264, Exam #2, Summer 2022 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. Find the limit of the infinite series. $\sum_{n=1}^{\infty}\frac{4}{n(n+2)}$ (10 points)

$\frac{A}{n}+\frac{B}{n+2} \Rightarrow A(n+2)+Bn=4$

$An+2A+Bn=4$

$2A=4 \quad A=2$

$A+B=0 \quad B=-2$

$$\sum_{n=1}^{\infty}\frac{2}{n}-\frac{2}{n+2}=\frac{2}{1}+\frac{2}{2}-\lim_{n\to\infty}\underbrace{\frac{2}{n+1}+\frac{2}{n+2}}_{\to 0}$$

$$=\boxed{3}$$

2. Determine the convergence or divergence of each series. (6 points each)

a. $\sum_{n=1}^{\infty} ne^{-n}$

converges $\int_1^{\infty} ne^{-n}\,dn=\lim_{b\to\infty}\underbrace{-(b+1)e^{-b}}_{\to 0}+(2)e^{-1}$ converges

by L'Hopital's

integral test

ratio test also works $\lim_{n\to\infty}\frac{(n+1)e^{-(n+1)}}{n\,e^{-n}}=(1)\cdot\frac{1}{e}<1$

b. $\sum_{n=2}^{\infty}\frac{1}{(n-1)^{4/5}}$

diverges

by p test

$p<1$

replace $n-1$ w/ k

$\sum_{k=1}^{\infty}\frac{1}{k^{4/5}}$

c. $\sum_{n=1}^{\infty}\frac{1}{4\sqrt[3]{n}-1}$

limit comparison $\lim_{n\to\infty}\frac{\frac{1}{4\sqrt[3]{n}-1}}{\frac{1}{4\sqrt[3]{n}}}=\frac{4\sqrt[3]{n}}{4\sqrt[3]{n}-1}=\frac{1}{1-\frac{1}{4\sqrt[3]{n}}}=1$

$\sum\frac{1}{4\sqrt[3]{n}}$ diverges by p-test $p=1/3<1$

d. $\sum_{n=1}^{\infty}\frac{5n-3}{n^2-2n+5}$

limit comparison

$$\lim_{n\to\infty}\frac{\frac{5n-3}{n^2-2n+5}}{\frac{5}{n}}=\frac{1-\frac{3}{5n}}{1-\frac{2}{n}-\frac{5}{n^2}}=1$$

$\sum\frac{5}{n}$ diverges by p test, $p=1$

e. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ Converges by alternating series test

$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$

f. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\frac{2}{3}\right)^n}{n^2}$ converges by ratio test $\lim_{n\to\infty} \frac{(2/3)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(2/3)^n} =$

$\lim_{n\to\infty} \underbrace{\left(\frac{n^2}{(n+1)^2}\right)}_{\to 1} \cdot \left(\frac{2}{3}\right) < 1$

g. $\sum_{n=1}^{\infty} \left(\frac{-2n}{3n+1}\right)^{2n}$ root test $\lim_{n\to\infty} \sqrt[n]{\left(\frac{-2n}{3n+1}\right)^{2n}} = \lim_{n\to\infty} \frac{4n^2}{(3n+1)^2} = \lim_{n\to\infty} \frac{4n^2}{9n^2+6n+1}$

$= \frac{4}{9} < 1$ converges

3. For the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$, if the series converges, does it converge conditionally or absolutely?

Find the partial sum of the first six terms and then state the maximum error on the sum at that term. (16 points)

Converges absolutely since $\sum \frac{1}{n^2}$ converges.

$\frac{1}{1} - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} = \frac{973}{1200} \approx 0.810833$ max error $= \frac{1}{49}$

≈ 0.020408

4. Find N such that $R_N \le 0.001$ for the convergent series. (10 points)

$\sum_{n=1}^{\infty} \frac{1}{n^4} = \pi^4/90 \approx 1.08232$ $N=7 \Rightarrow 1.08154$

error $= 0.00078$

7 terms are needed if you know true sum

estimate w/integral test $\int_N^{\infty} \frac{1}{n^4}\,dn = \frac{-1}{3}n^{-3}\Big|_N^{\infty} = \frac{1}{3N^3} \le 0.001$ $N^3 = \frac{1000}{3}$ $N = 6.9336\ldots$ $N=7$

5. Use the first four terms of the Taylor series to approximate $\int_0^1 e^{-x^2}\,dx$. (10 points)

$e^x = \sum \frac{x^n}{n!}$

$e^{-x^2} = \sum \frac{(-x)^{2n}}{n!} = \sum \frac{(-1)^n x^{2n}}{n!}$

$\int_0^1 1 - \frac{x^2}{1} + \frac{x^4}{2} - \frac{x^6}{6}\,dx$

$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\Big|_0^1 = \frac{26}{35} \approx 0.742857$

6. For the sequence below. i) Determine if the sequence is monotonic (or is monotonic after some finite value of n). You may determine this graphically or by calculating derivatives. ii) Determine the bounds (above and below of the sequence). iii) Can you apply the bounded & monotonic theorem for convergence to this sequence? iv) Does this sequence converge by another theorem? If so, which one? v) If the sequence converges, what does it converge to? (20 points)

$$a_n = ne^{-n/2}$$

ii) bounded above by 2, below (>0) by 0

iii) for $n>2$, yes. we can apply bounded & monotonic

$\therefore$ converges.

i) maximum at $n=2$ after that it is monotonic decreasing

iv) N/A

v) converges to 0 $\lim_{n\to\infty} ne^{-n/2} = 0$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

7. Find an expression for the nth partial sum of $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$. (12 points)

$$S_n = 2 + 1 + \frac{-2}{n+1} - \frac{2}{n+2}$$

8. For each of the series in #2, state the test used to determine convergence. (9 points each)

a. $\sum_{n=1}^{\infty} ne^{-n}$ integral or ratio

b. $\sum_{n=2}^{\infty} \frac{1}{(n-1)^{4/5}}$ p-test

c. $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$ limit comparison + p-test

d. $\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$ limit comparison + p-test

e. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ alternating series test

f. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\frac{2}{3}\right)^n}{n^2}$ ratio test

g. $\sum_{n=1}^{\infty} \left(\frac{-2n}{3n+1}\right)^{2n}$ root test

9. Determine the radius of convergence for the series $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$. State the interval of convergence and clearly indicate whether it is open, closed or half-open. (10 points)

$\lim_{n\to\infty} \left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = \lim_{n\to\infty}\left[\frac{n+2}{n+1}\right]^{1} \cdot \lim_{n\to\infty} |x| < 1$ when $|x|<1$

$(-1,1)$ both endpoints are not included since $n+1$ in the limit does not go to 0.

10. Find the nth Taylor polynomial centered at the given c. Use the included table to show work. (15 points)

$g(x) = \cos 4x, n = 6, c = \frac{\pi}{2}$

n	n!	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	$\cos 4x$	1	$(x-\pi/2)^0=1$	1
1	1	$-4\sin 4x$	0	$(x-\pi/2)^1$	0
2	2	$-16\cos 4x$	-16	$(x-\pi/2)^2$	$-\frac{16}{2}(x-\pi/2)^2$
3	6	$64\sin 4x$	0	$(x-\pi/2)^3$	0
4	24	$256\cos 4x$	256	$(x-\pi/2)^4$	$\frac{256}{24}(x-\pi/2)^4$
5	120	$-1024\sin 4x$	0	$(x-\pi/2)^5$	0
6	720	$-4096\cos 4x$	-4096	$(x-\pi/2)^6$	$-\frac{4096}{720}(x-\pi/2)^6$

$$P_n(x) = 1 - 8(x-\pi/2)^2 + \frac{32}{3}(x-\pi/2)^4 - \frac{256}{45}(x-\pi/2)^6$$

11. Find a power series for the functions using the geometric series method. (20 points each)

a. $f(x) = \frac{3}{2x-1} = -\frac{3}{1-2x}$ $a=-3$ $r=2x$ $\boxed{\sum_{n=0}^{\infty} -3(2x)^n}$

$$\frac{a}{1-x} = a(1-x)^{-1}$$

$$-a(1-x)^{-2}(-1) = a(1-x)^{-2}$$

$$2a(1-x)^{-3}$$

$$6a(1-x)^{-4} = \frac{6a}{(1-r)^4}$$

$$\sum_{n=0}^{\infty} ax^n \to \sum_{n=1}^{\infty} anx^{n-1} \to$$

$$\sum_{n=2}^{\infty} an(n-1)x^{n-2} \to$$

$$\sum_{n=3}^{\infty} an(n-1)(n-2)x^{n-3} \to$$

$$\sum_{n=0} a(n+3)(n+2)(n+1)x^n$$

r^n

b. $f(x) = \frac{7x^3}{(1+3x^2)^4}$

$a = 7x^3$ $r = (-3x^2)$

$$\sum_{n=0}^{\infty} \frac{7}{6}x^3(n+3)(n+2)(n+1)(-3x^2)^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{7}{6}(-3)^n x^{2n+3}(n+3)(n+2)(n+1)}$$