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Review of Integration Techniques

Exponential and Log Integrals.

$$\frac{d}{dx}(e^x) = e^x$$

$$\int e^x dx = e^x + C$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

Recall that the power rule has an exception

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

What if the base of the exponential or log is not e? Recall $\ln(x) = \log_e(x)$

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{(\ln a)x}$$

$$\int \frac{1}{(\ln a)x} dx = \log_{\mathrm{a}} x + C$$

Example where we would have (want) to use the log-base-a rather than natural log is with a substitution.

$$\int \frac{\log_a x}{x} dx$$

$$u = \log_a x\,, du = \frac{1}{x}\left(\frac{1}{\ln a}\right) dx, \ln a\,du = \frac{1}{x} dx$$

$$\int \ln a\, u du = \frac{1}{2}u^2(\ln a) + C = \frac{1}{2}\ln a\ \ (\log_a x)^2 + C$$

Change-base of rule:

$$\log_a(x) = \frac{\ln x}{\ln a}$$

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$$

Review of exponential and log algebra rules

$$e^a e^b = e^{a+b}$$

$$e^{x+\ln x} = e^x e^{\ln x} = e^x(x)$$

$$\log(MN) = \log M + \log N$$

$$\frac{e^a}{e^b} = e^{a-b}$$

$$\log\left(\frac{M}{N}\right) = \log M - \log N$$

$$(e^a)^b = e^{ab}$$

$$\log M^r = r \log M$$

Example.

$$\int x e^{x^2} dx$$

When you have an exponential function, if u-substitution is going to work, the exponent of the exponential is typically u.

$$u = x^2, du = 2xdx, \frac{1}{2}du = xdx$$

$$\int (e^{x^2})(xdx) = \int \frac{1}{2}\ e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$$

For logarithms, you may see problems that have logarithms in the integral. If that is the case, then the log must be the substitution. We can take a derivative of the log, we can't integrate.

Logs also come up when we have rational expressions.

$$\frac{kf'(x)}{f(x)}$$

$$\int \frac{3x}{x^2+1}dx = 3\int \frac{1}{u}\left(\frac{1}{2}\right)dx = \frac{3}{2}\int \frac{1}{u}du = \frac{3}{2}\ln|u| + C = \frac{3}{2}\ln(x^2+1) + C$$

$$u = x^2 + 1, du = 2xdx, \frac{1}{2}du = xdx$$

Some things to keep in mind about rational functions generally:

- If the numerator is the same degree or bigger than the denominator, you need to do long division.
- If you have the sum of squares in the denominator (and no x in the numerator) think about arctangent function.
- The exception to the power rule that produces a lnx only applies to 1/x, not any other power of x in the denominator.

Recall $\int \tan x\, dx = -\ln|\cos x| + C = \ln|\sec x| + C$, similarly $\int \cot x\, dx = \ln|\sin x| + C$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\, dx = -\ln|\cot x + \csc x| + C$$

$$\frac{d}{dx}[\ln|\sec x + \tan x|] = \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x) = \frac{\sec(x)(\tan x + \sec x)}{\sec x + \tan x} = \sec x$$

Exponential growth and decay

$$P(t) = P_0 e^{kt}$$

If k>0, it's a growth problem, if k<0, then it's a decay problem.

A compounding problem for finance $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

How do the problems in this section differ from precalculus?

- Derivation of formulas uses some calculus (can prove that $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$)
- Sometimes the problems ask about rates: what is the (instantaneous) rate of change of a growth problem at a point in time. Take the derivative to find the rate.
- Sometimes a problem might ask about the accumulated interest. Integrate over the given period time using the rate function to obtain the accumulated interest.

Hyperbolic Trig Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

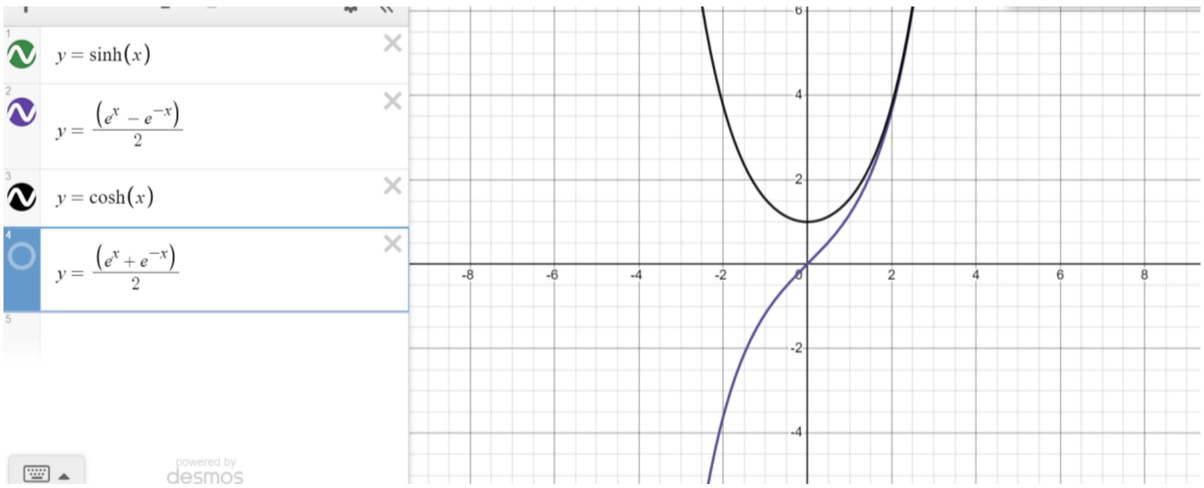
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Compare to regular (circular) trig functions.

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$e^{ix} = \cos(x) + i\sin(x)$$



$$\tan(x) = \frac{\sin x}{\cos x}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth(x) = \frac{1}{\tanh(x)} = \frac{\cosh(x)}{\sinh(x)}$$

$$\text{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$

$$\text{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$$

Pythagorean identities are not based on circles, but on hyperbolas.

$$\cosh^2(x) - \sinh^2(x) = 1$$

Instead of $\cos^2 x + \sin^2 x = 1$

Derivatives and antiderivatives

Derivatives/antiderivatives mostly follow the regular trig derivative rules except for two sign changes.

$$\frac{d}{dx}(\sinh(x)) = \cosh(x)$$

$$\frac{d}{dx}(\cosh(x)) = \sinh(x)$$

$$\frac{d}{dx}(\tanh(x)) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\coth(x)) = -\operatorname{csch}^2(x)$$

$$\frac{d}{dx}(\operatorname{sech}(x)) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\int \sinh(x)\,dx = \cosh(x) + C$$

$$\int \cosh(x)\,dx = \sinh(x) + C$$

$$\int \tanh(x)\,dx = \ln|\cosh x| + C$$

$$\int \coth x\,dx = \ln\,|\sinh\,(x)| + C$$

$$\int \operatorname{sech}^2 x\,dx = \tanh(x) + C$$

Etc.

The cosh(x) function works well in the arc length formula.

Suppose we want to find the length of arc for a curve given by $y = \cosh\,(x)$.

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}dx = \int_a^b \sqrt{1 + [\sinh(x)]^2}dx = \int_a^b \sqrt{\cosh^2 x}\,dx = \int_a^b \cosh(x)\,dx$$
$$= \sinh(b) - \sinh(a)$$

Review of integration techniques

- It's a good idea to memorize your basic derivative rules
- Is there algebra I can do to simplify this problem?
    - Exponential and log rules?
    - You may be able to apply an identity (trig identity) to simplify. (power reducing identities and Pythagorean identities)

    - Rational expressions may need long division to proceed
    - If you have an expression like $\frac{1}{1-e^x}$ (or $\frac{1}{1-e^{-x}}$) but multiply by the exponential with the opposite sign in the exponent (top and bottom).
- U-substitution – undoes the chain, you a composite multiplied by a simpler function that is like the derivative of the "inside" function.
- Change of variables