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Integration by Parts (3.1)
Trigonometric Integrals (3.2)

Integration by Parts (essentially the reverse of product rule)

$$(uv)' = u'v + v'u$$
$$\int (uv)' = \int u'v + \int uv'$$
$$uv = \int u'v + \int uv'$$

$$uv - \int u'v = \int uv'$$

Integration by parts

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

$$\int udv = uv - \int vdu$$

The idea is that the new integral is easier to integrate than the original.

$$\int xe^x dx$$

General rule:
Select u so that it becomes easier when you take the derivative and/or select v' to be something that may stay the same or get easier when you integrate.

$$u = x, dv = e^x dx$$
$$du = dx, v = \int e^x dx = e^x$$

$$\int udv = uv - \int vdu$$

$$xe^x - \int e^x dx = xe^x - e^x + C$$

$$f(x) = xe^x - e^x + C$$

$$f'(x) = e^x + xe^x - e^x = xe^x$$

LIATE – as a sequence for choosing u
Logarithms – Inverse Trig Functions – Algebraic – Trigonometric – Exponential functions
Algebraic – polynomials vs. everything else (roots, rational expressions)

$$\int x^3 \ln x \, dx$$

$$u = \ln x, dv = x^3 dx$$
$$du = \frac{1}{x} dx, v = \frac{1}{4} x^4$$

$$uv - \int v du = \frac{1}{4} x^4 \ln x - \int \frac{1}{4} x^4 \left(\frac{1}{x} dx\right) = \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 dx = \frac{1}{4} x^4 \ln x - \frac{1}{4}\left(\frac{1}{4} x^4\right) + C$$

$$\frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + C$$

Integration by parts can be used to integrate $\int \ln x \, dx$, and $\int \arcsin x \, dx$

$$\int \arcsin x \, dx$$

$$u = \arcsin x, dv = dx$$
$$du = \frac{1}{\sqrt{1 - x^2}} dx, v = x$$

$$x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} dx$$

$$w = 1 - x^2, dw = -2x dx, -\frac{1}{2} dw = x dx$$
$$\int \frac{x}{\sqrt{1 - x^2}} dx = \int -\frac{1}{2} w^{-\frac{1}{2}} dw = -w^{\frac{1}{2}}$$

$$x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} dx = x \arcsin x - \left(-\sqrt{1 - x^2}\right) + C$$

$$x \arcsin x + \sqrt{1 - x^2} + C$$

Change of variable vs. integration by parts

$$\int x\sqrt{x - 2} dx$$

Change of variables. Let $u = \sqrt{x - 2}, u^2 = x - 2, u^2 + 2 = x, 2u du = dx$

$$\int x\sqrt{x - 2} dx = \int (u^2 + 2) u (2u du) = \int 2u^4 + 4u^2 du = \frac{2}{5} u^5 + \frac{4}{3} u^3 + C$$

$$\frac{2}{5} (x - 2)^{\frac{5}{2}} + \frac{4}{3} (x - 2)^{\frac{3}{2}} + C$$

$$2(x-2)^{\frac{3}{2}}\left(\frac{1}{5}(x-2)+\frac{2}{3}\right)+C=\frac{2}{15}(x-2)^{\frac{3}{2}}(3(x-2)+10)+C=\frac{2}{15}(x-2)^{\frac{3}{2}}(3x+4)+C$$

Integration by parts

$$u=x, dv=(x-2)^{\frac{1}{2}}dx$$
$$du=dx, v=\frac{2}{3}(x-2)^{\frac{3}{2}}$$

$$\frac{2}{3}x(x-2)^{\frac{3}{2}}-\int\frac{2}{3}(x-2)^{\frac{3}{2}}dx=\frac{2}{3}x(x-2)^{\frac{3}{2}}-\frac{2}{3}\left(\frac{2}{5}(x-2)^{\frac{5}{2}}\right)+C$$

$$\frac{2}{3}x(x-2)^{\frac{3}{2}}-\frac{4}{15}(x-2)^{\frac{5}{2}}+C=\frac{2}{15}(x-2)^{\frac{3}{2}}\big(5x-2(x-2)\big)+C=\frac{2}{15}(x-2)^{\frac{3}{2}}(3x+4)+C$$

Tabular method

$$\int x^4\sin x\,dx$$

Long way

$$u=x^4, dv=\sin x\,dx$$
$$du=4x^3dx, v=-\cos x$$

$$-x^4\cos x-\int -4x^3\cos x\,dx=-x^4\cos x+\int 4x^3\cos x\,dx$$

$$u=4x^3, dv=\cos x\,dx$$
$$du=12x^2dx, v=\sin x$$

$$-x^4\cos x+4x^3\sin x-\int 12x^2\sin x\,dx$$

$$u=12x^2, dv=\sin x\,dx$$
$$du=24xdx, v=-\cos x$$

$$-x^4\cos x+4x^3\sin x+12x^2\cos x-\int 24x\cos x\,dx$$

$$u=24x, dv=\cos x\,dx$$
$$du=24dx, v=\sin x$$

$$-x^4\cos x+4x^3\sin x+12x^2\cos x-24x\sin x-\int 24\sin x\,dx$$

$$-x^4\cos x+4x^3\sin x+12x^2\cos x-24x\sin x-24\cos x+C$$

Sign	u	dv
+	x^4	$\sin x$
-	$4x^3$	$-\cos x$
+	$12x^2$	$-\sin x$
-	$24x$	$\cos x$
+	24	$\sin x$
-	0	$-\cos x$

$$-x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24x\sin x - 24\cos x + C$$

The u function needs to be something that will eventually go to zero (polynomial), and the dv function has to be something that can be integrated multiple times on its own (sinx, cosx, e^x, etc. roots with linear expressions under the root) that does not need to "borrow" or doesn't simplify with the u function.

Integration by parts with substitution (borrowing…)

$$\int x^3 e^{x^2}\,dx = \int (x^2)xe^{x^2}\,dx$$

$$u = x^2, dv = xe^{x^2}dx$$
$$du = 2xdx, v = \frac{1}{2}e^{x^2}$$

$$\frac{1}{2}x^2e^{x^2} - \int xe^{x^2}dx = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$$

Looping integrals.

$$\int e^x \sin x\,dx$$

$$u = \sin x\,, dv = e^x dx$$
$$du = \cos x\,dx, v = e^x$$

$$e^x\sin x - \int e^x\cos x\,dx$$

$$u = \cos x\,, dv = e^x dx$$
$$du = -\sin x\,dx, v = e^x$$

$$e^x\sin x - \left[e^x\cos x - \int -e^x\sin x\,dx\right] = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$$

$$\int e^x\sin x\,dx = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$$

Add the integral to both sides

$$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x\,dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + C$$

Trigonometric Integrals

Main technique is to rewrite integrals involving trig functions so that we can simplify the expressions, or apply identities to be able to do a u-substitution.

Main identities: definitions in terms of sine and cosine, the power reducing identities, the Pythagorean identities.

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x + \sin^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$

Three general cases:
Combinations of sine and cosine
Combinations of tangent and secant
Combinations of other stuff (including 3 or more trig functions).

Combinations of sine and cosine.

$$\int \sin^m x \cos^n x\,dx$$

Hardest case (because of the algebra) is if m and n are both even. Apply the power reducing identity. Do algebra, maybe apply again. Until you have only linear trig functions.

$$\int \sin^2 x \cos^2 x\,dx = \int \frac{1}{2}(1 - \cos 2x)\frac{1}{2}(1 + \cos 2x)dx = \frac{1}{4}\int 1 - \cos^2 2x\,dx =$$

$$\frac{1}{4}\int 1 - \frac{1}{2}(1 + \cos 4x)dx = \frac{1}{4}\int 1 - \frac{1}{2} - \frac{1}{2}\cos 4x\,dx = \frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos 4x\,dx = \frac{1}{8}\int 1 - \cos 4x\,dx =$$

$$\frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right] + C$$

If one power is odd…
Then the odd power, pull out one copy to set aside for doing substitution, and replace the remaining even powers using the pythagorean identity.

$$\int \sin^4 x \cos^3 x\,dx = \int \sin^4 x \cos^2 x\,(\cos x\,dx) = \int \sin^4 x\,(1 - \sin^2 x) \cos x\,dx$$

$$\int (\sin^4 x - \sin^6 x) \cos x \, dx$$

$$u = \sin x \,, du = \cos x \, dx$$

$$\int u^4 - u^6 du = \frac{1}{5}u^5 - \frac{1}{7}u^7 + C = \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$$

Be careful with $\frac{1}{\sin x} \neq \sin^{-1} x$

Write $\frac{1}{\sin x} = (\sin x)^{-1}$

Negative powers and fractional powers all work as long as one power is odd. Then follow this method.

Combinations of tangent and secant

$$\int \tan^m x \sec^n dx$$

1. Odd powers of secant (with even powers of tangent)

$$\int \sec^3 x \, dx$$

   This integral needs to be by parts, and it loops.

$$u = \sec x \,, dv = \sec^2 x \, dx$$
$$du = \sec x \tan x \,, v = \tan x$$

$$\sec x \tan x - \int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec x \,(\sec^2 x - 1) dx$$

$$\sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x \, dx$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x \, dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$$

2. Even powers of secant
   a. Both are even

$$\int \sec^4 x \tan^4 x \, dx$$

Pull out 2 copies of secant ($\sec^2 x$) convert the remaining secants to tangents.

$$\int (1 + \tan^2 x)(\tan^4 x) \sec^2 x \, dx$$

$$u = \tan x \,, du = \sec^2 x \, dx$$

$$\int (\tan^4 x + \tan^6 x) \sec^2 x \, dx = \int u^4 + u^6 du = \frac{1}{5}u^5 + \frac{1}{7}u^7 + C = \frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C$$

b. Powers of secant and odd tangents

$$\int \sec^3 x \tan^3 x \, dx$$

Pull out one copy of secant and one copy of tangent to be du and convert the remaining even powers of tangent to secants.

$$\int \sec^2 x \tan^2 x \,(\sec x \tan x) dx = \int \sec^2 x \,(\sec^2 x - 1)(\sec x \tan x) dx$$

$$u = \sec x \,, du = \sec x \tan x$$

$$\int u^4 - u^2 du = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

3[rd] case: everything else
If all else fails, use identities to switch everything to sine and cosine, to simplify and then follow the sine/cosine rules.

Next time: trig substitution

Review $u = \sin x$, what is $\cos x$ in terms of u?