6/22/2022

Trig Substitution (3.3)
Partial Fractions (3.4)

Trig Substitution is a method for integrating problems involving $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$
The substitutions align with the expressions in the inverse trig function derivatives. (all involve a square root).

Consider $\int \sqrt{9 - x^2}dx$
How do we do this algebraically?

For this form $\sqrt{a^2 - x^2}$ form, we make a substitution of $x = a \sin\theta$.

For our problem $a = 3$, so $x = 3\sin\theta$.

$$\sqrt{9 - x^2} = \sqrt{9 - (3\sin\theta)^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9(1 - \sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

$$dx = 3\cos\theta\, d\theta$$

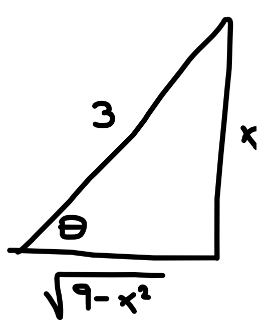
$$\int \sqrt{9 - x^2}dx = \int 3\cos\theta\,(3\cos\theta)\, d\theta = 9\int \cos^2\theta\, d\theta = \frac{9}{2}\int 1 + \cos 2\theta\, d\theta =$$

$$\frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

Go back to x

$$x = 3\sin\theta$$
$$\frac{x}{3} = \sin\theta$$
$$\arcsin\left(\frac{x}{3}\right) = \theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$



$$\cos\theta = \frac{\sqrt{9 - x^2}}{3}$$

$$\frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C = \frac{9}{2}\left[\arcsin\left(\frac{x}{3}\right) + \frac{1}{2}2\sin\theta\cos\theta\right] + C = \frac{9}{2}\left[\arcsin\left(\frac{x}{3}\right) + \frac{x}{3}\left(\frac{\sqrt{9 - x^2}}{3}\right)\right] + C$$

$$\frac{9}{2}\arcsin\left(\frac{x}{3}\right)+\frac{1}{2}x\sqrt{9-x^2}+C$$

Integrate $\int \frac{1}{\sqrt{1+x^2}}dx$.

Form here is $\sqrt{a^2+x^2}$ and so we are going to substitute $x=a\tan\theta$
In this case $x=\tan\theta$

$$\sqrt{1+x^2}=\sqrt{1+\tan^2\theta}=\sqrt{\sec^2\theta}=\sec\theta$$

$$dx=\sec^2\theta\,d\theta$$

$$\int\frac{1}{\sqrt{1+x^2}}dx=\int\frac{1}{\sec\theta}(\sec^2\theta)d\theta=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C$$

$$\ln|\sec\theta+\tan\theta|+C=\ln\left|\sqrt{1+x^2}+x\right|+C$$

Integrate $\int\sqrt{x^2-9}dx$

Form here is $\sqrt{x^2-a^2}$, so the form of substitution is $x=a\sec\theta$
In this problem $x=3\sec\theta$

$$\sqrt{x^2-9}=\sqrt{(3\sec\theta)^2-9}=\sqrt{9\sec^2\theta-9}=\sqrt{9(\sec^2\theta-1)}=\sqrt{9\tan^2\theta}=3\tan\theta$$

$$dx=3\sec\theta\tan\theta\,d\theta$$

$$\int\sqrt{x^2-9}dx=\int 3\tan\theta\,(3\sec\theta\tan\theta)d\theta=9\int\tan^2\theta\sec\theta\,d\theta=$$

$$9\int(\sec^2\theta-1)\sec\theta\,d\theta=9\int\sec^3\theta-\sec\theta\,d\theta$$

$$\int\sec^3\theta\,d\theta=\frac{1}{2}(\sec\theta\tan\theta+\ln|sec\,\theta+tan\,\theta|)+C$$

$$9\int\sec^3\theta-\sec\theta\,d\theta=9\left[\frac{1}{2}(\sec\theta\tan\theta+\ln|sec\,\theta+tan\,\theta|)-\ln|sec\,\theta+tan\,\theta|\right]+C=$$

$$9\left[\frac{1}{2}(\sec\theta\tan\theta-\ln|sec\,\theta+tan\,\theta|)\right]+C=\frac{9}{2}\left[\frac{x}{3}\left(\frac{\sqrt{x^2-9}}{3}\right)-\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|\right]+C$$

$$x=3\sec\theta$$

$$\frac{x}{3} = \sec\theta$$

$$\sqrt{x^2 - 9} = 3\tan\theta$$

$$\frac{\sqrt{x^2 - 9}}{3} = \tan\theta$$

Integrate $\int \frac{1}{\sqrt{x^2+4x-12}} dx$

$$x^2 + 4x - 12 = x^2 + 4x + 4 - 4 - 12 = x^2 + 4x + 4 - 16 = (x+2)^2 - 16$$

$$\int \frac{1}{\sqrt{x^2 + 4x - 12}} dx = \int \frac{1}{\sqrt{(x+2)^2 - 16}} dx$$

$$u = x + 2 = a\sec\theta = 4\sec\theta$$

$$x = 4\sec\theta - 2$$
$$dx = 4\sec\theta\tan\theta\, d\theta$$

$$\sqrt{x^2 + 4x - 12} = \sqrt{(x+2)^2 - 16} = \sqrt{(4\sec\theta)^2 - 16} = \sqrt{16\sec^2\theta - 16} = \sqrt{16(\sec^2\theta - 1)} =$$

$$\sqrt{16\tan^2\theta} = 4\tan\theta$$

$$\int \frac{1}{\sqrt{(x+2)^2 - 16}} dx = \int \frac{1}{4\tan\theta} 4\sec\theta\tan\theta\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x+2}{4} + \frac{\sqrt{x^2 + 4x - 12}}{4}\right| + C$$

$$\tan\ \theta = \frac{\sqrt{x^2 + 4x - 12}}{4}$$

$$\frac{x+2}{4} = \sec\theta$$

Half-powers are where this technique is needed. It can be done for whole powers, but it’s not necessary. But ½ powers, 3/2 powers, 5/2 powers, etc. or their negative.

The other three functions can also be used, but they introduce an extra minus sign. Cosine can be used instead of sine (but for the negative it is the same). And cotangent in place of tangent, and cosecant in place of secant, but for the negative signs.

Partial Fractions

Is a method of decomposing a fraction into smaller pieces when its denominator can be factored.

(is only applied when the numerator is a lower degree than the denominator).

Integrate.

$$\int \frac{3x}{x^2 - x - 2}dx = \int \frac{3x}{(x-2)(x+1)}dx$$

What we want is to find coefficients A and B such that

$$\frac{3x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$A(x+1) + B(x-2) = 3x$$

Traditional way of solving for A and B:

$$Ax + A + Bx - 2B = 3x$$

Collect the coefficients of x and the constants

$$Ax + Bx = 3x \; or \; A + B = 3$$
$$A - 2B = 0$$

$$A = 2B$$
$$2B + B = 3$$
$$3B = 3$$
$$B = 1$$
$$A = 2$$

Alternate way of solving for A and B.
Take advantage of the factors that multiple A and B, and pick a value of x that makes one disappear.

$$A(x+1) + B(x-2) = 3x$$

Let x=2

$$A(3) + B(0) = 6$$
$$3A = 6$$
$$A = 2$$

Let x=-1

$$A(0) + B(-3) = -3$$
$$-3B = -3$$
$$B = 1$$

Where this runs into snags is where there is no x that will make the factor zero (unfactorable quadratics)

$$\int \frac{3x}{x^2 - x - 2}dx = \int \frac{2}{x-2} + \frac{1}{x+1}dx = 2\ln|x-2| + \ln|x+1| + C$$

The general rules for partial fractions:

If you have a linear factor in the denominator of the form $(x-c)$ with no powers, then decompose that into $\frac{A}{x-c}$.

If you have a linear factor that is repeated (raised to a power), then decompose that into one term for each power up to the power in the expression

Decompose $(x-c)^3$ into terms like $\frac{A}{x-c}+\frac{B}{(x-c)^2}+\frac{C}{(x-c)^3}$

If a factor is a non-factorable quadratic ax^2+bx+c then decompose the fraction into $\frac{Ax+B}{ax^2+bx+c}$
For instance, a factor like (x^2+1) would produce a term like $\frac{Ax+B}{x^2+1}$

If the quadratic factor is repeated, then we follow the scheme for linear factors except that the numerators are linear.

If a factor was $(x^2+1)^2$ we would decompose that into $\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$

Let's look at a complicated scenario

$$\frac{x^3+6x^2-11}{(x-2)^2(x+3)(x^2+4)x^3}$$

Use partial fractions to decompose.

$$\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+3}+\frac{Dx+E}{x^2+4}+\frac{F}{x}+\frac{G}{x^2}+\frac{H}{x^3}$$

Integrate $\int \frac{x-2}{(2x-1)^2(x-1)}dx$

Decompose

$$\int \frac{x-2}{(2x-1)^2(x-1)}dx=\int \frac{A}{2x-1}+\frac{B}{(2x-1)^2}+\frac{C}{x-1}dx$$

$$A(2x-1)(x-1)+B(x-1)+C(2x-1)^2=x-2$$

X=1

$$A(1)(0)+B(0)+C(1)^2=-1$$
$$C=-1$$

X=1/2

$$A(0)\left(-\frac{1}{2}\right)+B\left(-\frac{1}{2}\right)+C(0)^2=-\frac{3}{2}$$

$$-\frac{1}{2}B=-\frac{3}{2}$$

$$B=3$$

X=0

$$A(-1)(-1)+(3)(-1)+(-1)(-1)^2=-2$$
$$A-3-1=-2$$
$$A-4=-2$$
$$A=2$$

$$\int \frac{2}{2x-1}+\frac{3}{(2x-1)^2}+\frac{-1}{x-1}dx=\ln|2x-1|-\frac{3}{2(2x-1)}-\ln|x-1|+C$$

$$\int \frac{3}{(2x-1)^2}dx$$
$$u=2x-1$$
$$du=2dx$$
$$\frac{1}{2}du=dx$$

$$\int \frac{3}{(2x-1)^2}dx=\int 3\frac{\left(\frac{1}{2}\right)du}{u^2}=\int \frac{3}{2}u^{-2}du=-\frac{3}{2}u^{-1}=-\frac{3}{2(2x-1)}$$

Integrate.

$$\int \frac{2x-3}{x^3+x}dx=\int \frac{2x-3}{x(x^2+1)}dx=\int \frac{A}{x}+\frac{Bx+C}{x^2+1}dx$$

$$A(x^2+1)+(Bx+C)x=2x-3$$

X=0

$$A(1)+(B(0)+C)(0)=-3$$
$$A=-3$$

X=1

$$-3(2)+(B+C)(1)=-1$$
$$B+C=5$$

X=-1

$$(-3)(2)+(-B+C)(-1)=-5$$
$$B-C=1$$

$$B+C=5$$
$$B-C=1$$

$$2B=6, B=3, C=2$$

$$\int \frac{-3}{x}+\frac{3x+2}{x^2+1}dx=\int -\frac{3}{x}+\frac{3x}{x^2+1}+\frac{2}{x^2+1}dx$$

$$-3\ln x+\frac{3}{2}\ln(x^2+1)+2\arctan x+C$$

Suppose I had $\int \frac{\cos x}{\sin^2 x - \sin x} dx$

Let $u = \sin x \,, du = \cos x \, dx$
Rewrite the integral $\int \frac{1}{(u^2 - u)} du$
Factor the denominator, do partial fractions. Don't forget to go back to x at the end.

Review factoring techniques: sum and difference of cubes, factor by grouping (4 terms)
Review long division with polynomials

Next week: finish Chapter 3 (numerical, improper, tables) and review for exam (review Monday, exam Wednesday)