6/8/2022

Work
Probability, Center of Mass

Work
The general idea is that W=Fd, if the force and the distance are fixed: they are the same everywhere. So we want to deal with these situations where either the force or the distance varies as we do the work.

$$W = \int_a^b F(x)dx$$

Or

$$W = \int_a^b \frac{dF}{dx}(xdx)$$

Spring example. (similar examples use gravity or charges particles.)
The spring force is Hooke's Law. $F = kx$ (in some problems, they may give you a function with a cubic term).

Similarly, gravity uses $F = \frac{k}{x^2}$. Inverse square law.

In a typical spring problem:
A force of 16 N is applied to a spring whose natural length is 1 m, and is used to stretch the spring by 20 centimeters from its natural length. Find the word done by stretching the spring an additional 10 centimeters.

$$16 = k(.20)$$
$$k = 80$$

$$W = \int_{0.20}^{0.30} 80xdx = 40x^2|_{.2}^{.3} = 40[0.3^2 - 0.2^2] = 2$$

2 Newton-meters

In English/imperial units, use feet rather than inches. Work units will be foot-pounds. In English unit problems, the "weight" is the force. Mass is "slugs". If you see mass, you need to convert to force: so in SI in problem kg has to be multiplied by gravity.

For a gravity problem the math is mostly similar except that the distance is measured from the center of gravity. If you are on the surface of the Earth, that is x when the problem gives you weight.

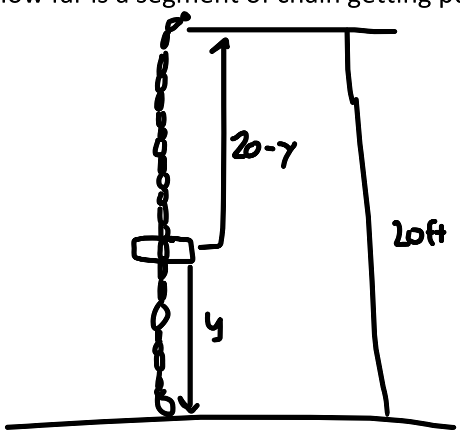
Chain problem (hanging chains)
The general idea here is (in most cases) that we are winding up a chain with a given weight per unit length.

We need to come with two pieces of information: what is the weight of a segment of the chain, and how far is that segment of the chain getting moved?

The weight section of the formula is the weight density given in the problem.

We have a 20 foot chain hanging down to the ground that weighs 4 pounds per foot. How much work is done in winding up the chain?

Density = 4 pounds/ft. Force = 4dy
How far is a segment of chain getting pulled? How far does it travel? (20-y)



$$W = \int_0^{20} 4(20 - y)dy = \int_0^{20} 4ydy$$

The second one is if you start with y at the top instead of y=0 on the ground.

$$\int_0^{20} 4(20 - y)dy = 4\left[20y - \frac{1}{2}y^2\right]_0^{20} = 4[400 - 200] = 800ft - lbs.$$

Tank problems.
If we have a tank of liquid that we want to drain or pump out, the force (the weight of the slice we are moving out of the tank) is going to depend a great deal on the geometry of the tank itself. In some cases the slices will all be the same size (rectangular base or a cylinder) and in some cases, the size of the slices will change as we go up or down the tank (pyramids, cones or spheres), but they will change in a regular way (we'll need an equation that relates the size of the slice to the height). And the liquid will have some density measure.

The force is the weight of the slice of liquid being moved: density times the volume of the slice
Volume: area times the thickness of the slice (dy)
Distance traveled of the slice.
Integrate force times the distance.

We have a cylindrical tank filled with water. The tank has a height of 20 feet and a diameter of 12 feet. The density of water is 62.4 lbs/ft^3. (radius = 6ft) Find the work done to pump the water over the top of the tank.

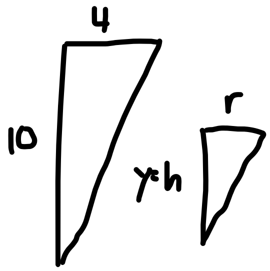
$$W = \int_0^{20} 62.4(\pi\, 6^2)(20 - y)dy = 62.4\pi(36)\left[20y - \frac{1}{2}y^2\right]_0^{20} = 62.4(\pi)(36)(200) =$$

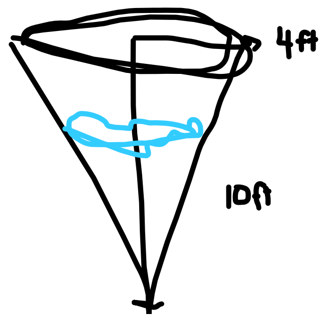
$$1{,}452{,}169.8\ ft-lbs.$$

What about cones and pyramids and spheres?

Use the idea of similar triangles.

We have a conical tank (with the point at the bottom), with a height of 10 feet and a radius at the top of 4 feet. The tank is half-full. We want to find the work done pumping all the water in tank out over the top.





$$\frac{10}{4}=\frac{y}{r}$$

$$\frac{4}{10}=\frac{r}{y}$$

$$\frac{10}{y}=\frac{4}{r}$$

Solve for r to get the radius (of our cylindrical slice) to be in terms of the height from the bottom of the tank.

$$10r=4y$$
$$r=\frac{4}{10}y=\frac{2}{5}y$$

The area of the cylindrical slice is $A=\pi r^2=\pi\left(\frac{2}{5}y\right)^2=\frac{4\pi}{25}y^2$

Volume = $\frac{4\pi}{25}y^2dy$

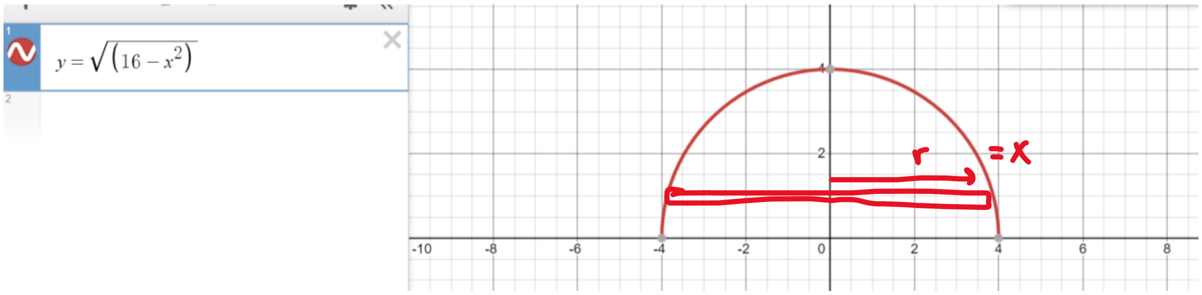
Weight = density times the volume = 64.2 ($\frac{4\pi}{25}y^2dy$)

$$W=\int_0^5 64.2\left(\frac{4\pi}{25}\right)y^2(10-y)dy=64.2\left(\frac{4\pi}{25}\right)\int_0^5 10y^2-y^3dy=$$

$$64.2\left(\frac{4\pi}{25}\right)\left[\frac{10}{3}y^3-\frac{1}{4}y^4\right]_0^5=64.2\left(\frac{4\pi}{25}\right)\left[\frac{1250}{3}-\frac{625}{4}\right]=8403.8\ ft-lbs.$$

A pyramid is similar but the cross-sectional slices are squares usually (but they depend on y)

In the case of a hemisphere or sphere, think about the circle as the cross-section.



$$x^2 + y^2 = r^2$$
$$r = x = \sqrt{16 - y^2}$$

Area of the slice $\pi r^2 = \pi\left(\sqrt{16 - y^2}\right)^2 = \pi(16 - y^2)$

Centers of Mass

We want to be able to find the geometric center of mass, assuming that the density of the object is constant.

The total mass: the density of the region (by area) the area of the region.
The area of the region is just our standard integral. Typically, these problems use ρ as the density.

$$M = \rho \int_a^b f(x) - g(x)dx$$

To find the geometric mass point in the x-direction, we multiply the integral on the inside by x.

$$M_y = \rho \int_a^b x[f(x) - g(x)]dx$$

The moment of mass from the y-axis.
The center of mass along the x-axis is $\bar{x} = \frac{M_y}{M}$

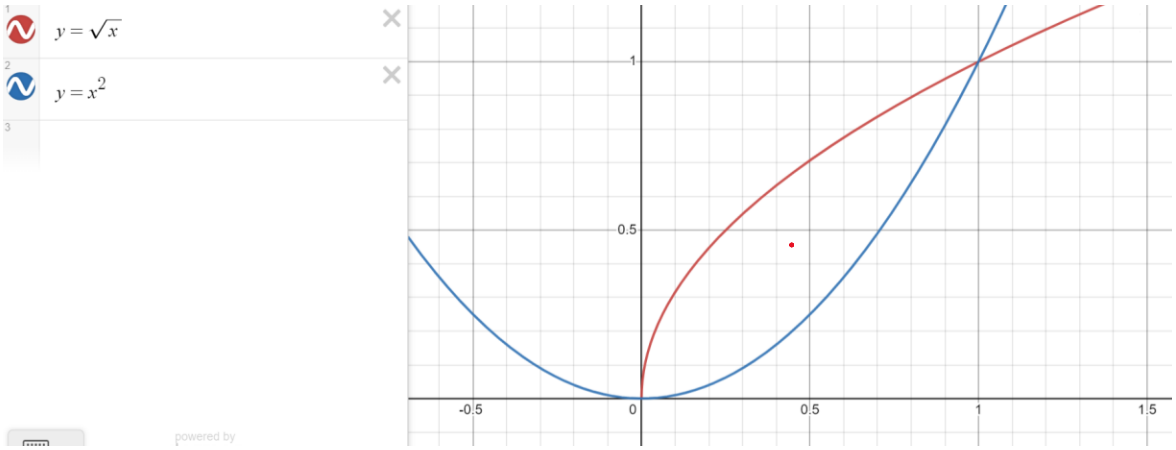
To find the geometric mass point in the y-direction, we actually square both functions and multiply by ½.

$$M_x = \frac{\rho}{2} \int_a^b [f(x)]^2 - [g(x)]^2 dx$$

This is the moment of mass from the x-axis.
The center of mass along the y-axis is $\bar{y} = \frac{M_x}{M}$

The center of mass = centroid $(\bar{x}, \bar{y})$.

Find the centroid of the region bounded by $y = \sqrt{x}, y = x^2$ (with constant density).



$$M = \int_0^1 \sqrt{x} - x^2\,dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$M_y = \int_0^1 x(\sqrt{x} - x^2)dx = \int_0^1 x^{\frac{3}{2}} - x^3 dx = \frac{2}{5}x^{\frac{5}{2}} - \frac{1}{4}x^4\Big|_0^1 = \frac{2}{5} - \frac{1}{4} = \frac{3}{20}$$

$$M_x = \frac{1}{2}\int_0^1 \left(\sqrt{x}\right)^2 - (x^2)^2 dx = \frac{1}{2}\int_0^1 x - x^4 dx = \frac{1}{2}\left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{20}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{3}{20}}{\frac{1}{3}} = \frac{3}{20} \times \frac{3}{1} = \frac{9}{20}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{3}{20}}{\frac{1}{3}} = \frac{3}{20} \times \frac{3}{1} = \frac{9}{20}$$

$$\left(\frac{9}{20}, \frac{9}{20}\right)$$

(symmetry is uncommon).

Probability.
In statistics, continuous probability distributions can be treated with calculus. In a valid probability distribution, the area under the curve is equal 1.

Suppose I have a function that I want to make a probability distribution. $f(x) = Kx^2, 0 \leq x \leq 4$
I need to find a constant K, such that the area under the curve is equal to 1.

$$P(X) = \int_0^4 Kx^2 dx = 1$$

$$K\int_0^1 x^2 dx = K\left(\frac{1}{3}x^3\right]_0^4 = \frac{K}{3}(64) = 1$$

$$\frac{64}{3}K = 1$$

$$K = \frac{3}{64}$$

Probability distribution is $f(x) = \frac{3}{64}x^2, 0 \leq x \leq 4$

What is the probability that x is less than 1?

$$P(X \leq 1) = \int_0^1 \frac{3}{64}x^2 dx = \frac{1}{64}x^3\Big|_0^1 = \frac{1}{64}$$

What is the mean of this distribution?
We multiply the function under the integral by x to get the average (mean)

$$\bar{x} = \int_0^4 x\left(\frac{3}{64}x^2\right)dx = \int_0^4 \frac{3}{64}x^3 dx = \frac{3}{64}\left(\frac{1}{4}x^4\right]_0^4 = \frac{3}{64}\left(\frac{1}{4}\right)(4^4) = 3$$

In statistics, the most common distribution.

$$f(x) = Ke^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}$$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}$$