MTH 263, Homework #1, Summer 2023 Name KEY

Instructions: Write your work up neatly and attach to this page. Record your final answers (only) directly on this page if they are short; if too long indicate which page of the work the answer is on and mark it clearly. Use exact values unless specifically asked to round.

1. Numerically estimate the limits.

a. $\lim_{x\to 2} \frac{x-2}{x^2-4}$ $\frac{\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

b. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

c. $\lim_{x\to -5} \frac{\sqrt{4-x}-3}{x+5} \cdot \frac{\sqrt{4-x}+3}{\sqrt{4-x}+3} = \lim_{x\to -5} \frac{\overset{-\cancel{(x+5)}}{4-x-9}}{\cancel{(x+5)}(\sqrt{4-x}+3)} = \lim_{x\to -5} \frac{-1}{\sqrt{4-x}+3} = \frac{-1}{6}$

d. $\lim_{x\to 1} \frac{x^4-1}{x^6-1}$ $\frac{\cancel{(x^2-1)}(x^2+1)}{\cancel{(x^2-1)}(x^4+x^2+1)} = \frac{2}{3}$

2. Use a graph to determine the value of the limit, if it exists.

a. $\lim_{x\to 2} \frac{|x-2|}{x-2}$ DNE

b. $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ DNE

c. $f(x) = \begin{cases} -\frac{1}{(x+2)^2}, & x<0 \\ x, & 0\le x\le 2 \\ -\frac{1}{2}x^2+4, & x>2 \end{cases}$

i. $\lim_{x\to 0^-} f(x)$ $-1/4$

ii. $\lim_{x\to 0^+} f(x)$ 0

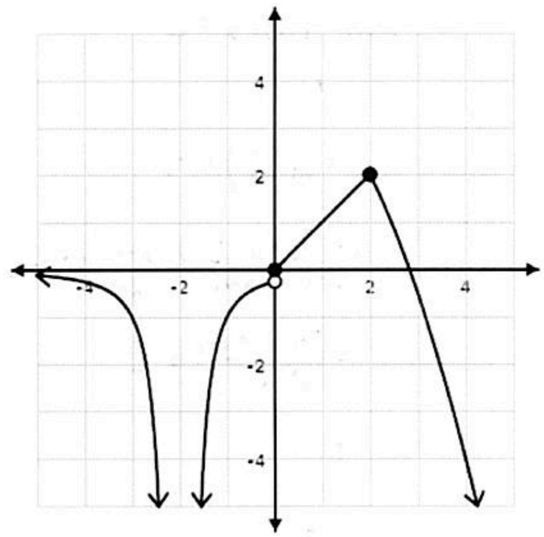
iii. $\lim_{x\to 0} f(x)$ DNE

iv. $\lim_{x\to 2^-} f(x)$ 2

v. $\lim_{x\to 2^+} f(x)$ 2

vi. $\lim_{x\to 2} f(x)$ 2

vii. $\lim_{x\to -2} f(x)$ $-\infty$



d. $f(x) = \begin{cases} x^2-1, & x<1 \\ \ln(2-x), & 1\le x<2 \\ -\frac{1}{x-2}, & x>2 \end{cases}$

i. $\lim_{x\to 1^-} f(x)$ 0

ii. $\lim_{x\to 1^+} f(x)$ 0

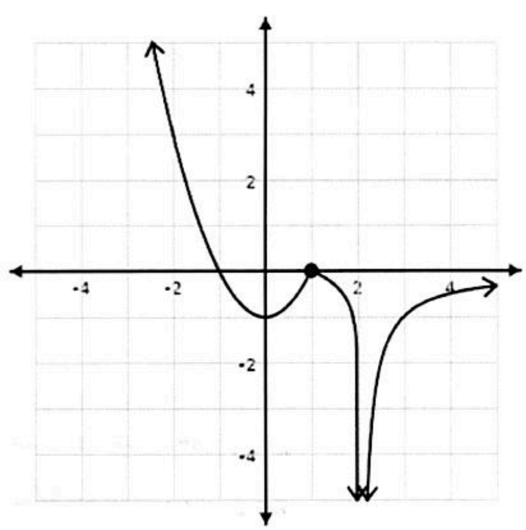
iii. $\lim_{x\to 1} f(x)$ 0

iv. $\lim_{x\to 2^-} f(x)$ $-\infty$

v. $\lim_{x\to 2^+} f(x)$ $-\infty$

vi. $\lim_{x\to 2} f(x)$ $-\infty$

e. $\lim_{x\to 1} \sqrt[3]{x}\ln|x-2| = 0$



3. Consider $f(x) = \begin{cases} \sin(x), & x < 0 \\ 1 - \cos(x), & 0 \le x \le \pi \\ \cos(x), & x > \pi \end{cases}$. Are there any values for c for which $\lim\limits_{x\to c} f(x)$ does not exist?

$c = \pi$

$\lim\limits_{x\to\pi^-} f(x) = 2 \quad \lim\limits_{x\to\pi^+} f(x) = -1$

4. Find the limit L: $\lim\limits_{x\to 4}(4 - \frac{x}{2})$. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - 4| < \delta$.

pg 4

5. Find the limit L. Then use the $\varepsilon - \delta$ definition of the limit to prove it. pg 4

a. $\lim\limits_{x\to -3}(2x + 5)$

b. $\lim\limits_{x\to 1}(x^2 + 1)$

c. $\lim\limits_{x\to 1}\left(\frac{2}{3}x + 9\right)$

d. $\lim\limits_{x\to 3}|x - 3|$

6. Find the limit algebraically using properties of limits.

a. $\lim\limits_{x\to 1}(3x^3 - 4x^2 + 3)$

$3(1) - 4(1) + 3 = 2$

b. $\lim\limits_{x\to 0}\sec 2x$ $\quad 1$

c. $\lim\limits_{x\to 1}\ln\left(\frac{x}{e^x}\right) = \ln\left(\frac{1}{e}\right) = -1$

d. $\lim\limits_{x\to 1}\frac{x^3 - x}{x - 1}$ $\quad \frac{(x^2-1)x}{x-1} = \lim\limits_{x\to 1}\frac{x\cancel{(x-1)}(x+1)}{\cancel{x-1}} = 2$

e. $\lim\limits_{x\to 0}\frac{e^{2x} - 1}{e^x - 1}$ $\quad \frac{\cancel{(e^x-1)}(e^x+1)}{\cancel{e^x-1}} = 2$

f. $\lim\limits_{x\to 0}\frac{\sin^2 x}{x}$ $\quad 0$

g. $\lim\limits_{x\to 0}\frac{\sin 2x}{\sin 3x}$ $\quad \frac{2}{3}$

h. $\lim\limits_{x\to 1}\frac{2x - 3}{x + 5}$ $\quad \frac{-1}{6}$

i. $\lim\limits_{x\to 0}e^{-x}\sin \pi x$ $\quad 0$

j. $\lim\limits_{x\to 0}\frac{-x^2 + 3x}{x}$ $\quad \frac{\cancel{x}(3-x)}{\cancel{x}} = 3$

k. $\lim\limits_{x\to 2}\frac{x^3 - 8}{x - 2}$ $\quad \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{x-2}} = 12$

l. $\lim\limits_{x\to 0}\frac{\sqrt{3+x} - \sqrt{3}}{x} \cdot \frac{\sqrt{3+x} + \sqrt{3}}{\sqrt{3+x} + \sqrt{3}} = \lim\limits_{x\to 0}\frac{\cancel{3}+x-\cancel{3}}{x(\sqrt{3+x} + \sqrt{3})} = \frac{1}{2\sqrt{3}}$

m. $\lim\limits_{x\to 0}\frac{1 - e^{-x}}{e^x - 1} \cdot \frac{e^x}{e^x} = \lim\limits_{x\to 0}\frac{\cancel{e^x - 1}}{e^x\cancel{(e^x - 1)}} = 1$

n. $\lim\limits_{x\to 0}\frac{\sin 3x}{x}$ $\quad 3$

7. Use the squeeze theorem to find the limits.

a. $\lim\limits_{x\to 0}x\cos x$

b. $\lim\limits_{x\to 0}x\cos\frac{1}{x}$ $\quad$ pg 5

8. Find the limit if it exists.

a. $\lim\limits_{x\to 2}f(x), f(x) = \begin{cases} x^2 - 4x + 6, & x < 2 \\ -x^2 + 4x - 2, & x \ge 2 \end{cases}$

b. $\lim\limits_{x\to 6^-}\ln(6 - x)$

c. $\lim\limits_{x\to 1^+}f(x), f(x) = \begin{cases} x, & x < 1 \\ 1 - x, & x \ge 1 \end{cases}$ $\quad$ pg 5

9. Find any points of discontinuity. Is the discontinuity removable or not?

a. $f(x) = \frac{3}{x-2}$ $\quad x = 2$. not removable

b. $f(x) = \frac{|x-3|}{x-3}$ $\quad x = 3$ not removable

c. $f(x) = \frac{x}{x^2+1}$ $\quad$ no discontinuities

d. $f(x) = \begin{cases} -2x, & x \le 2 \quad -4 \\ x^2 - 4x + 1, & x > 2 \quad -3 \end{cases}$

$x = 2$

not removable

10. Find the value of a that makes the function continuous. $g(x) = \begin{cases} \frac{4\sin x}{x}, x < 0 \\ a - 2x, x \geq 0 \end{cases}$

$a - 2\overset{=0}{x} = 4$

$a = 4$

11. Find the vertical asymptotes, if they exist.

a. $f(x) = \frac{4}{(x-2)^3}$ $X = 2$

b. $f(x) = \frac{-3x^3+12x+9}{x^4-3x^3-x+3}$

$x^3(x-3) - (x-3) = (x^3-1)(x-3)$

$(x-1)(x^2+x+1)(x-3)$

$X = 1, x = 3$

12. Find the limit.

a. $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2-x}}$ $= 1$

b. $\lim_{x\to\infty} \frac{8}{4-10^{-\frac{x}{2}}}$ $= 2$

c. $\lim_{x\to\infty} \frac{x^2+3}{2x^2-1}$ $= \frac{1}{2}$

4. $\lim_{x\to 4}\left(4-\frac{x}{2}\right) = 4-\frac{4}{2} = 4-2 = 2$

$|4-\frac{x}{2}-2| < 0.01 \rightarrow |2-\frac{x}{2}| = |\frac{x}{2}-2| = \frac{1}{2}|x-4| < 0.01 \rightarrow |x-4| < 0.02$

$\delta = 0.02$

5a. $\lim_{x\to -3} 2x+5 = -6+5 = -1$

$|2x+5+1| < \varepsilon \rightarrow |2x+6| < \varepsilon \rightarrow 2|x+3| < \varepsilon \rightarrow |x+3| < \varepsilon/2 \qquad \delta = \varepsilon/2$

Proof. if $|x+3| < \delta$ and if $\delta = \varepsilon/2$ then $|x+3| < \varepsilon/2 \rightarrow 2|x+3| < \varepsilon \rightarrow |2x+6| < \varepsilon \rightarrow$ $|2x+5-(-1)| < \varepsilon$ $\therefore$ the $\lim_{x\to -3} 2x+5 = -1$

b. $\lim_{x\to 1}(x^2+1) = 2$

$|x^2+1-2| < \varepsilon \rightarrow |x^2-1| < \varepsilon \rightarrow |(x-1)(x+1)| < \varepsilon \rightarrow |x-1|\cdot|x+1| < \varepsilon$

on an interval around $x=1$ such as $[0,2]$ $\quad x+1 < 3 \rightarrow |x-1|\cdot 3 < \varepsilon$ near $x=1$

$|x-1| < \varepsilon/3 = \delta$

Proof.

if $|x-1| < \delta$ and suppose $\delta = \varepsilon/3$ then $|x-1| < \varepsilon/3 \rightarrow 3|x-1| < \varepsilon$ since $x+1 < 3$ for all values of x on $[0,2]$ around $x=1$, this implies $|x+1|\cdot|x-1| < \varepsilon \rightarrow |x^2-1| < \varepsilon$ $\rightarrow |x^2+1-2| < \varepsilon$. Therefore, $\lim_{x\to 1}(x^2+1) = 2$

c. $\lim_{x\to 1}\left(\frac{2}{3}x+9\right) = \frac{2}{3}+9 = \frac{29}{3}$

$|\frac{2}{3}x+9-\frac{29}{3}| < \varepsilon \rightarrow |\frac{2}{3}x-\frac{2}{3}| < \varepsilon \rightarrow \frac{2}{3}|x-1| < \varepsilon \rightarrow |x-1| < \frac{3}{2}\varepsilon = \delta$

Proof. if $|x-1| < \delta$ and if we let $\delta = \frac{3}{2}\varepsilon$ then $|x-1| < \frac{3}{2}\varepsilon \rightarrow \frac{2}{3}|x-1| < \varepsilon \rightarrow |\frac{2}{3}x-\frac{2}{3}| < \varepsilon$ $\rightarrow |\frac{2}{3}x+9-\frac{29}{3}| < \varepsilon$ $\therefore$ $\lim_{x\to 1}\frac{2}{3}x+9 = \frac{29}{3}$

d. $\lim_{x\to 3}|x-3| = 0 \qquad |x-3| < \varepsilon \rightarrow \varepsilon = \delta$

Proof. if $|x-3| < \delta$ and suppose $\delta = \varepsilon$, then $|x-3-0| < \varepsilon$ $\therefore$ $\lim_{x\to 3}|x-3| = 0$.

7.a. $\lim_{x\to 0} x\cos x = 0$

$$-1 < \cos x < 1$$

$$-x < x\cos x < x$$

$$\lim_{x\to 0} -x < \lim_{x\to 0} x\cos x < \lim_{x\to 0} x$$

$$0 < \lim_{x\to 0} x\cos x < 0$$

b. $\lim_{x\to 0} x\cos\frac{1}{x} = 0$

$$-1 < \cos\left(\frac{1}{x}\right) < 1$$

$$-x < x\cos\left(\frac{1}{x}\right) < x$$

$$\lim_{x\to 0} -x < \lim_{x\to 0} x\cos\left(\frac{1}{x}\right) < \lim_{x\to 0} x$$

$$0 < \lim_{x\to 0} x\cos\left(\frac{1}{x}\right) < 0$$

8a. $\lim_{x\to 2^-} x^2-4x+6 = 4-8+6 = 10-8 = 2$

$\lim_{x\to 2^+} -x^2+4x-2 = -4+8-2 = 2$

$\therefore \lim_{x\to 2} f(x) = 2$

b. $\lim_{x\to 6^-} \ln(6-x) = -\infty$

c. $\lim_{x\to 1^+} 1-x = 0$