MTH 265, Exam #3, Summer 2024 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. Find the curvature of the curve $\vec{r}(t) = t^3\hat{\imath} + \ln t\,\hat{\jmath} + \tan^{-1} t\,\hat{k}$ at the point $t = 1$. What is the radius of curvature at the same point? (12 points)

$$r' = 3t^2\hat{\imath} + \frac{1}{t}\hat{\jmath} + \frac{1}{1+t^2}\hat{k} \qquad r'(1) = 3\hat{\imath} + 1\hat{\jmath} + \frac{1}{2}\hat{k}$$

$$r'' = 6t\hat{\imath} - \frac{1}{t^2}\hat{\jmath} + \frac{-2t}{(1+t^2)^2}\hat{k} \qquad r''(1) = 6\hat{\imath} - 1\hat{\jmath} - \frac{1}{2}\hat{k}$$

$$r' \times r'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 1 & 1/2 \\ 6 & -1 & -1/2 \end{vmatrix} = \underbrace{\left(-\tfrac{1}{2}+\tfrac{1}{2}\right)}_{=0}\hat{\imath} - \underbrace{\left(\tfrac{3}{2}-3\right)}_{+9/2}\hat{\jmath} + \underbrace{(-3-6)}_{-9}\hat{k}$$

$$\frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\sqrt{\frac{81}{4}+81}}{\left(\sqrt{9+1+\frac{1}{4}}\right)^3} = \frac{\sqrt{\frac{405}{4}}}{\frac{41\sqrt{41}}{8}} = \frac{4\sqrt{405}}{41\sqrt{41}} = K(1)$$

$$R(1) = \frac{1}{K(1)} = \frac{41\sqrt{41}}{4\sqrt{405}}$$

2. Find the area of the surface given by the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$. (8 points)

$$x^2+y^2 = (x^2+y^2)^2 = 4(x^2+y^2)$$

$$r^2 + r^4 = 4r^2$$

$$r^4 - 3r^2 = 0$$

$$r^2(r^2-3) = 0$$

$$r = 0,\ r = \pm\sqrt{3}$$

$$\int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2r}{\sqrt{4-r^2}}\,dr\,d\theta$$

$$u = 4-r^2$$

$$du = -2r\,dr$$

$$-\int u^{-1/2}\,du$$

$$-2u^{1/2}$$

$$\int_0^{2\pi} -2\sqrt{4-r^2}\,\Big|_0^{\sqrt{3}}\,d\theta$$

$$\cancel{-}2\int_0^{2\pi} \underbrace{1-2}_{-1}\,d\theta = 2\pi\cdot 2 = 4\pi$$

$$x^2+y^2+(z-2)^2 = 4$$

$$(z-2)^2 = 4-x^2-y^2$$

$$z-2 = \sqrt{4-x^2-y^2}$$

$$z = 2+\sqrt{4-x^2-y^2}$$

$$G = 2-z+\sqrt{4-x^2-y^2}$$

$$\nabla G = \left\langle \frac{-x}{\sqrt{4-x^2-y^2}}, \frac{-y}{\sqrt{4-x^2-y^2}}, -1 \right\rangle$$

$$\|\nabla G\| = \sqrt{\frac{x^2}{4-x^2-y^2} + \frac{y^2}{4-x^2-y^2} + 1} = \sqrt{\frac{4-\cancel{x^2}-\cancel{y^2}+\cancel{x^2}+\cancel{y^2}}{4-x^2-y^2}} = 2/\sqrt{4-x^2-y^2} = \frac{2}{\sqrt{4-r^2}}$$

3. Evaluate the flux integral $\int_S \int \vec{F} \cdot \vec{N} dS$ where $\vec{F} = 2z\hat{\imath} - 4x\hat{\jmath} + \hat{k}$ for the surface $S: z = 12 - 3x - 4y$ in the first octant. (12 points)

$G = 12 - 3x - 4y - z \qquad \langle -3, -4, -1 \rangle = \nabla G$ or $\langle 3, 4, 1 \rangle$

down $\qquad$ up

$F \cdot \nabla G = \langle 2z, -4x, 1 \rangle \cdot \langle 3, 4, 1 \rangle = 6z - 16x + 1$

$= 6(12 - 3x - 4y) - 16x + 1 = 72 - 18x - 24y - 16x + 1 = 73 - 34x - 24y$

$\int_0^4 \int_0^{3 - 3/4 x} 73 - 34x - 24y \, dy\, dx =$

$\frac{12 = 3x + 4y}{4}$

$3 = \frac{3}{4}x + y$

$y = 3 - 3/4 x$

$\int_0^4 73y - 34xy - 12y^2 \Big|_0^{3 - 3/4 x} dx =$

$\int_0^4 \frac{75}{4}x^2 - \frac{411}{4}x + 111 \, dx = \frac{25}{4}x^3 - \frac{411}{8}x^2 + 111x \Big|_0^4 = 22$

4. Use the divergence theorem to evaluate $\int_S \int \vec{F} \cdot \vec{N} dS$ for $\vec{F}(x, y, z) = (2xy - 1)\hat{\imath} + (3yz + 2)\hat{\jmath} + xz\hat{k}$ for the closed surface bounded by the cylinder $x^2 + y^2 = 4, z = 4$, and the coordinate planes. (14 points)

$\vec{\nabla} \cdot \vec{F} = 2y + 3z + x = 2r\sin\theta + 3z + r\cos\theta$

$\int_0^{\pi/2} \int_0^2 \int_0^4 2r^2 \sin\theta + 3rz + r^2\cos\theta \, dz\, dr\, d\theta =$

$\int_0^{\pi/2} \int_0^2 \Big[2r^2 \sin\theta z + \frac{3}{2} r z^2 + r^2 z \cos\theta \Big|_0^4 dr\, d\theta =$

$\int_0^{\pi/2} \int_0^2 8r^2 \sin\theta + 24r + 4r^2 \cos\theta \, dr\, d\theta =$

$\int_0^{\pi/2} \frac{8}{3} r^3 \sin\theta + 12r^2 + \frac{4}{3} r^3 \cos\theta \Big|_0^2 d\theta =$

$\int_0^{\pi/2} \frac{64}{3} \sin\theta + 48 + \frac{32}{3} \cos\theta \, d\theta = -\frac{64}{3}\cos\theta + 48\theta + \frac{32}{3}\sin\theta \Big|_0^{\pi/2}$

$= 24\pi + 32$

5. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ using Stokes' Theorem for $\vec{F}(x,y,z) = z\hat{\imath} + x\hat{\jmath} + xyz\hat{k}$ for $S: z = x^2 - y^2$, $0 \le x \le 1, 0 \le y \le 1$. (12 points)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & xyz \end{vmatrix} = (xz-0)\hat{\imath} - (yz-1)\hat{\jmath} + (1-0)\hat{k} = \langle xz, 1-yz, 1\rangle$$

$$G = -x^2+y^2+z \qquad \nabla G = \langle -2x, 2y, 1\rangle$$

$$(\vec{\nabla}\times\vec{F})\cdot\nabla G = -2x^2z + 2y - 2y^2\cancel{z} + 1 = -2x^2(x^2-y^2) - 2y^2(x^2-y^2) + 1 + 2y$$

$$= -2x^4 + \cancel{2x^2y^2} - \cancel{2x^2y^2} + 2y^4 + 1 + 2y \Rightarrow$$

$$\int_0^1\int_0^1 -2x^4+2y^4+1+2y\,dy\,dx = \int_0^1 \tfrac{2}{5}y^5 + y - 2x^4y\Big|_0^1 dx = \int_0^1 \tfrac{12}{5} - 2x^4\,dx = \tfrac{12}{5}x - \tfrac{2}{5}x^5\Big|_0^1 = 2$$

6. Write an integral for the arc length of the curve $\vec{r}(t) = \sqrt[3]{t^2}\hat{\imath} + \sqrt{t^3}\hat{\jmath} + t\hat{k}$ on the interval $[1,4]$. Evaluate it numerically. (6 points)

$t^{2/3} \quad t^{3/2}$

$$r'(t) = \tfrac{2}{3}t^{-1/3}\hat{\imath} + \tfrac{3}{2}t^{1/2}\hat{\jmath} + 1\hat{k}$$

$$\|r'(t)\| = \sqrt{\tfrac{4}{9}t^{-2/3} + \tfrac{9}{4}t + 1}$$

$$\int_1^4 \sqrt{\tfrac{4}{9}t^{-2/3} + \tfrac{9}{4}t + 1}\;dt \approx 7.7959\ldots$$

7. Find the directional derivative of the function $w = \arcsin xyz$ at the point $\left(1, \frac{1}{2}, 1\right)$ in the direction of $\vec{v} = \langle -1, 3, -2\rangle$. In what direction is the directional derivative a maximum? (10 points)

$$\hat{v} = \left\langle \tfrac{-1}{\sqrt{14}}, \tfrac{3}{\sqrt{14}}, \tfrac{-2}{\sqrt{14}} \right\rangle \qquad \|v\| = \sqrt{1+9+4} = \sqrt{14}$$

$$\nabla w = \left\langle \frac{yz}{\sqrt{1-x^2y^2z^2}}, \frac{xz}{\sqrt{1-x^2y^2z^2}}, \frac{xy}{\sqrt{1-x^2y^2z^2}} \right\rangle$$

$$\nabla w\left(1,\tfrac{1}{2},1\right) = \left\langle \frac{\frac{1}{2}}{\sqrt{1-1/4}}, \frac{1}{\sqrt{1-1/4}}, \frac{1/2}{\sqrt{1-1/4}} \right\rangle = \left\langle \tfrac{1}{2}\cdot\sqrt{\tfrac{4}{3}}, 1\cdot\sqrt{4/3}, \tfrac{1}{2}\sqrt{\tfrac{4}{3}} \right\rangle =$$

$$\left\langle \tfrac{1}{\sqrt{3}}, \tfrac{2}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} \right\rangle$$

$$\vec{\nabla} w\cdot\hat{v} = \left\langle \tfrac{1}{\sqrt{3}}, \tfrac{2}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} \right\rangle \cdot \left\langle -\tfrac{1}{\sqrt{14}}, \tfrac{3}{\sqrt{14}}, \tfrac{-2}{\sqrt{14}} \right\rangle =$$

$$\frac{-1}{\sqrt{42}} + \frac{6}{\sqrt{42}} - \frac{2}{\sqrt{42}} = \frac{3}{\sqrt{42}}$$

$$\max = \nabla w = \left\langle \tfrac{1}{\sqrt{3}}, \tfrac{2}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} \right\rangle$$

8. Use the Fundamental Theorem of Line Integrals to evaluate $\int_C (2xy+1)dx + (x^2 - y)dy$ on the line segment from $(0,2)$ *to* $(4,-2)$. (10 points)

$\varphi : \int 2xy + 1\, dx = x^2y + x + f(y)$

$\int x^2 - y\, dy = x^2y - \frac{1}{2}y^2 + g(x)$

$\varphi = x^2y + x - \frac{1}{2}y^2$

$\varphi(4,-2) - \varphi(0,2) = 16(-2) + 4 - \frac{1}{2}(4) - [0 + 0 - \frac{1}{2}(4)] =$

$-32 + 4 \cancel{-2} + \cancel{2} = -28$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

9. Find an equation of the tangent plane to the surface $\vec{r}(u,v) = u\cos v\,\hat{\imath} + u\sin v\,\hat{\jmath} + u\hat{k}$ at the point $u = 3, v = \frac{\pi}{4}$. (8 points)

$\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 3\right)$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (0 - u\cos v)\hat{\imath} - (0 - u\sin v)\hat{\jmath} + (u\cos^2 v + u\sin^2 v)\hat{k}$

$= -u\cos v\,\hat{\imath} - u\sin v\,\hat{\jmath} + u\hat{k}$

$\langle -\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 3 \rangle$

$-\frac{3}{\sqrt{2}}\left(x - \frac{3}{\sqrt{2}}\right) - \frac{3}{\sqrt{2}}\left(y - \frac{3}{\sqrt{2}}\right) + 3(z - 3) = 0$

10. Find the unit tangent vector for $\vec{r}(t) = (\sqrt{t^3} - 4)\hat{\imath} + (t^2 + 1)\hat{\jmath}$. (8 points)

$\vec{r}\,'(t) = \left(\frac{3}{2}t^{1/2}\right)\hat{\imath} + 2t\hat{\jmath}$

$\|r'(t)\| = \sqrt{\frac{9}{4}t + 4t^2}$

$\vec{T}(t) = \dfrac{\frac{3\sqrt{t}}{2}\hat{\imath} + 2t\hat{\jmath}}{\sqrt{\frac{9}{4}t + 4t^2}}$

11. Find the equation of the tangent plane to the surface $xyz + z^{2/3} = y^2$ at $(-1,1,1)$. (8 points)

$$F = xyz + z^{2/3} - y^2$$

$$\nabla F = \langle yz,\ xz - 2y,\ xy + \tfrac{2}{3}z^{-1/3} \rangle$$

$$\nabla F(-1,1,1) = \langle 1,\ -1-2,\ -1+\tfrac{2}{3} \rangle = \langle 1,\ -3,\ -\tfrac{1}{3} \rangle$$

plane:

$$1(x+1) - 3(y-1) - \tfrac{1}{3}(z-1) = 0$$

12. Use Green's Theorem to evaluate $\int_C \overset{M}{xy}dx + \overset{N}{x^2}dy$ on the path described by the boundary of the graphs $y = x^2, y = \sqrt{x}$ oriented counterclockwise. (12 points)

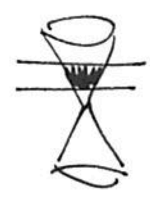
$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x = x$$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} x\, dy\, dx = \int_0^1 xy \Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 x^{3/2} - x^3\, dx = \frac{2}{5}x^{5/2} - \frac{1}{4}x^4 \Big|_0^1 =$$

$$\frac{2}{5} - \frac{1}{4} = \frac{3}{20}$$

13. Evaluate the surface integral $\int_S \int f(x,y,z)dS$ for $f(x,y,z) = x^2z^2$, S: on the cone $z^2 = x^2 + y^2$, between the planes $z = 1, z = 3$. [Hint: converting to cylindrical/polar will help.] (12 points)



$$z = r$$

$$z = \sqrt{x^2+y^2}$$

$$G = z - \sqrt{x^2+y^2}$$

$$\nabla G = \left\langle \frac{-x}{\sqrt{x^2+y^2}},\ \frac{-y}{\sqrt{x^2+y^2}},\ 1 \right\rangle$$

$$\|\nabla G\| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + \frac{1(x^2+y^2)}{x^2+y^2}} = \sqrt{\frac{2x^2+2y^2}{x^2+y^2}} = \sqrt{2}$$

$$\int_0^{2\pi}\int_1^3 \sqrt{2}\, r^2\cos^2\theta\, \underbrace{z^2}_{=r^2} r\, dr\, d\theta = \int_0^{2\pi}\int_1^3 \sqrt{2}\, r^5 \cos^2\theta\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{6} r^6 \Big|_1^3 \cos^2\theta\, d\theta =$$

$$\int_0^{2\pi} \frac{182\sqrt{2}}{3}(1+\cos 2\theta)\, d\theta = \frac{182\sqrt{2}}{3}\left[\theta + \tfrac{1}{2}\overset{0}{\cancel{\sin 2\theta}}\right]_0^{2\pi} = \frac{364\sqrt{2}}{3}\pi$$

Cylindrical

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$x^2 + y^2 = r^2$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$

Spherical

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$
$$x^2 + y^2 + z^2 = \rho^2$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$
$$\cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \phi$$
$$x^2 + y^2 = \rho^2\sin^2\phi = r^2$$

Dels

$$\frac{\partial}{\partial x} = \text{ partial derivative with respect to } x$$
$$\nabla f = grad\ f$$
$$\nabla^2 f = \nabla \cdot (\nabla f) = Laplacian\ of\ f$$
$$\nabla \cdot \vec{F} = div\ \vec{F}$$
$$\nabla \times \vec{F} = curl\ \vec{F}$$

Misc

$$ds = \|\vec{r}'(t)\| dt$$
$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{1}{R}$$
$$\vec{N} = \nabla F = \vec{r}_u \times \vec{r}_v$$